Pinning down QCD contributions to flavor anomalies

LIP Summer Internships 2022
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1 A crash course on the Standard Model

The Standard Model (SM) is a collection of gauge theories which describe the strong, electromagnetic and weak interactions. As illustrated in Fig. 1, a good deal of its physics content can be understood diagrammatically: The elementary fermions (quarks and leptons) carry certain types of charges (color, electric charge, ...), and they interact by exchanging gauge bosons. The precise way how these interactions work is determined by the **Feynman rules** which follow from the SM Lagrangian:

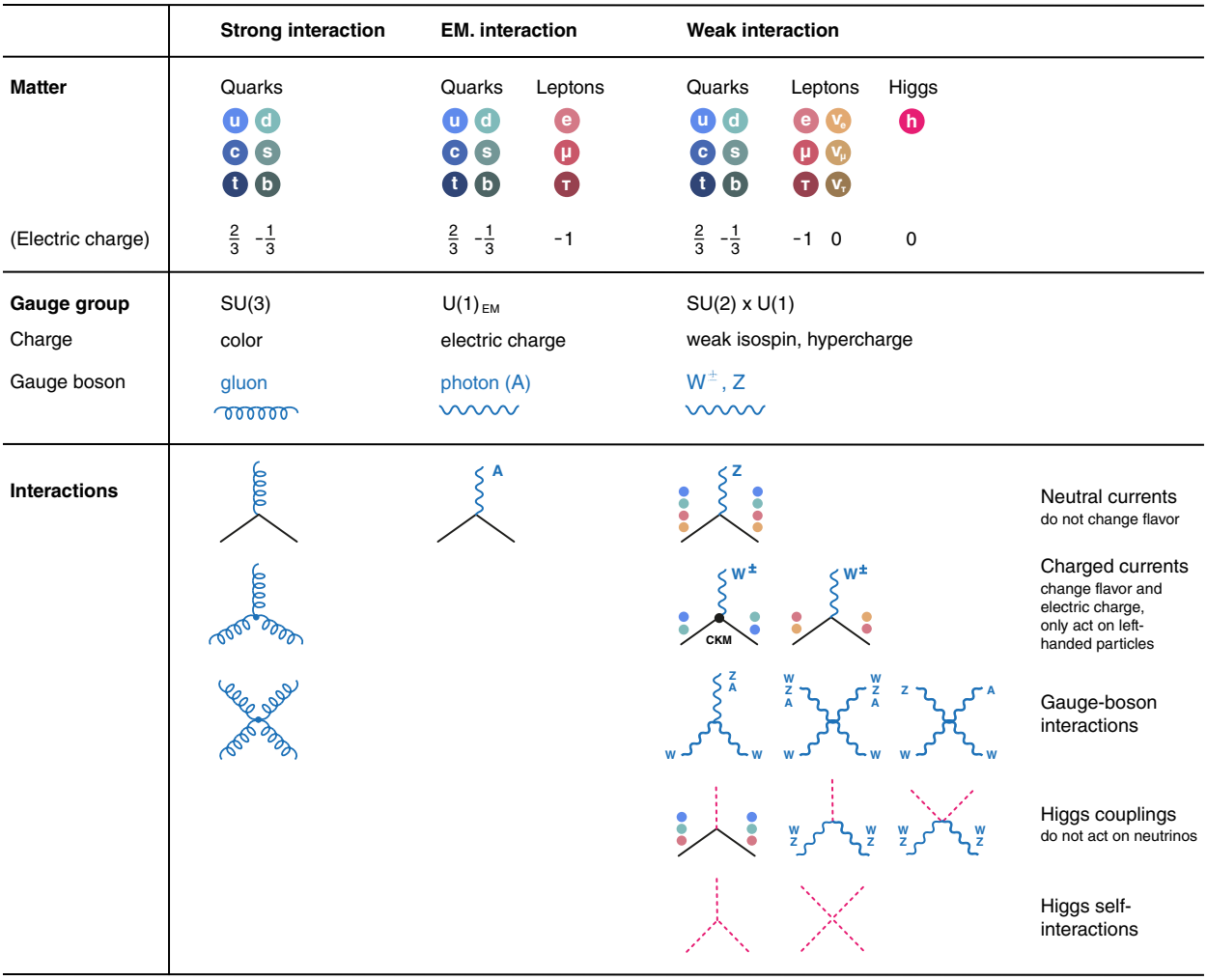
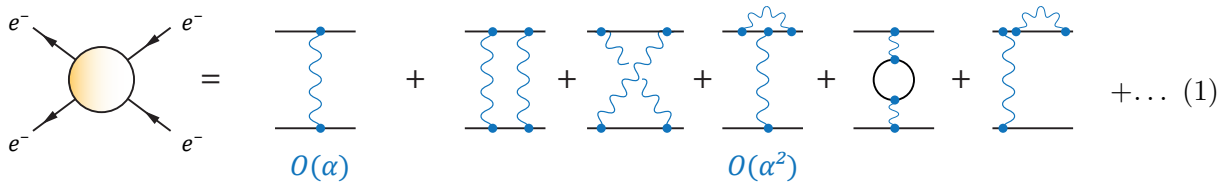


Figure 1: Standard Model

■ **Quantum Electrodynamics (QED)** describes the electromagnetic interaction. It tells us that particles carrying an electric charge interact with other electrically charged particles by exchanging massless photons. Take for example Møller scattering ($e^-e^- \to e^-e^-$):



(1)

Quantum field theory says that the process is the sum of all possible Feynman diagrams that can contribute. All such diagrams are constructed from only three elementary ingredients: the electron propagator (black solid line), the photon propagator (blue wavy line) and the electron-photon vertex. These objects have simple mathematical expressions, and each loop means that we integrate over the four-momentum. Ultimately, every process that can happen in QED can be constructed from these elementary rules, which are defined by the QED Lagrangian.

In particular, each vertex comes with a factor e (the electric charge), so we can order the diagrams with powers of the electromagnetic coupling $\alpha_{\text{QED}} = e^2/(4\pi) \approx 1/137$. Because this is a small quantity, we can apply **perturbation theory**: We expand any such process in powers of α_{QED} and stop the series after a few terms, because higher-order (loop) diagrams will be strongly suppressed. In fact, for many applications even the tree-level diagram (the first diagram on the r.h.s. above) is sufficient; this is basically what quantum mechanics describes, whereas quantum field theory also takes into account all possible quantum loop effects. The smallness of α_{QED} has led to the spectacular successes of QED in various precision measurements, from the anomalous magnetic moments of electrons and muons to the energy level splittings in atoms.

■ **Quantum Chromodynamics (QCD)** is the part of the SM describing the strong interaction, which acts on the color charge. Only quarks and gluons carry color charges, so they are the only particles that feel the strong interaction. QCD says that quarks interact with other quarks by exchanging massless gluons. However, in contrast to QED, which is a $U(1)$ gauge theory, QCD is a $SU(3)$ gauge theory, which is non-Abelian and this has important consequences: It implies that also the gluons themselves carry color charges, so they can interact with each other. For this reason the Feynman rules of QCD (first column in Fig. 1) not only comprise a quark-gluon vertex but also a three-gluon and a four-gluon vertex.

These additional interactions are also what cause the strong coupling α_{QCD} to strongly *run* with the momentum scale: At large energies, α_{QCD} is still small and quarks and gluons almost behave like free particles, which is called **asymptotic freedom**, so one can employ perturbation theory like in QED. At low momenta, on the other hand, the coupling becomes large (hence the name *strong* interaction), which implies that we can no longer expand processes in Feynman diagrams. If you replace the photons in Eq. (1) by gluons, then higher orders in α_{QCD} might be as large as lower orders and the series won't converge, so perturbation theory becomes useless. QCD is still completely specified by its Feynman rules, but we need **nonperturbative** methods to calculate things in practice.

A closely connected phenomenon is **confinement**: we cannot detect quarks and gluons as free particles because they are always confined in **hadrons**, which are bound states of quarks and gluons like mesons, baryons, etc. So even though we know perfectly well what the QCD Lagrangian and its elementary Feynman rules are, at the end of the day they are always folded into hadronic matrix elements. For example, we cannot directly measure how a photon couples to a quark but only how it couples to a hadron which consists of quarks and gluons (Fig. 2). This is what makes the study of the strong interaction a challenging and fascinating topic.

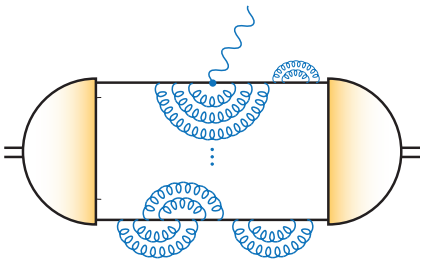


Figure 2: When a photon couples to a hadron (here a meson as a $q\bar{q}$ system), the quarks inside the hadron can exchange gluons in all possible ways. This cannot be calculated using perturbation theory so one needs nonperturbative methods.

Quarks also come in different **flavors** which form three generations: up and down, strange and charm, bottom and top. The different flavors come with different masses, from very light to very heavy:

$$m_u \sim m_d \sim 2 \ldots 6\,\text{MeV}, \qquad m_s \sim 100\ \text{MeV}, \qquad \begin{array}{ll} m_c \sim 1.3 & \text{GeV}, \\ m_b \sim 4.2 & \text{GeV}, \\ m_t \sim 173 & \text{GeV}. \end{array} \tag{2}$$

Since we cannot measure quarks directly, one has to define a convention what a 'quark mass' actually means; usually they are extracted from high-energy scattering experiments or through processes involving hadrons. Also here QCD plays a key role: For example, protons and neutrons consist of three light quarks (protons are made of uud and neutrons of ddu), but the mass of the proton is not $10 \ldots 15$ MeV, as you might expect by adding up three light quarks, but rather 940 MeV. In other words, 98% of the mass of the proton (and other hadrons too) must be generated by QCD itself. This is called **dynamical mass generation**, another non-perturbative effect that you won't get by simply adding up Feynman diagrams.

■ The **weak interactions** are the final piece of the SM. The SM combines the electromagnetic and weak interactions to an electroweak $SU(2) \times U(1)$ gauge theory, corresponding to weak isospin and hypercharge, which is spontaneously broken to $U(1)_{\text{EM}}$ by the Higgs mechanism. The Higgs boson, experimentally discovered in 2012, gives masses to the quarks, the charged leptons and the W and Z bosons, whereas the massless particle we know as the photon is a linear combination of one of the $SU(2)$ weak isospin gauge bosons and the $U(1)$ hypercharge gauge boson.

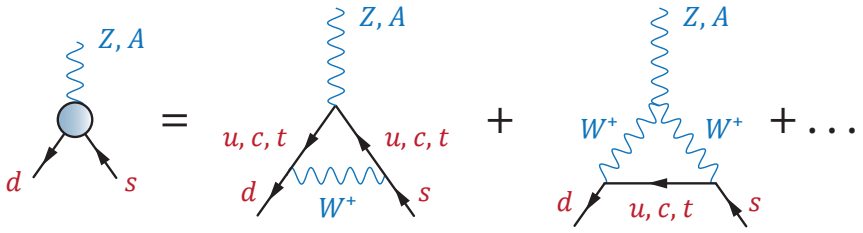
As you can see in Fig. 1, the Feynman rules for the weak interaction become quite messy. While the Z bosons couple to all fermions to produce neutral currents, the charged currents obtained by the $W^\pm$ bosons add or subtract one unit of electric charge so they must also change the flavor: electrons turn into neutrinos and vice versa, u quarks turn into d quarks and so on. This is the microscopic origin of neutron β decay, because a d quark inside a neutron (ddu) can turn into a u quark by emitting a W^- boson, thus creating a proton (uud), and the W^- then emits an electron and an antineutrino. When W bosons couple to quarks they can also change the generation, for example a u quark may turn into an s quark – these transitions are mediated by the off-diagonal elements in the **CKM matrix** which is attached to the vertex. Apart from the fermion-gauge-boson couplings, the gauge bosons also interact by themselves, and they couple to the Higgs (penultimate row in Fig. 1) which gives them a mass. The neutrinos and the photon do not couple to the Higgs and remain massless.

The electroweak sector is also a good place to look for physics beyond the SM – e.g., today we know that the neutrinos actually do have small masses, so the SM as it stands cannot be the full story. Also the experimentally observed flavor anomalies primarily show up through the electroweak interactions.

2 Flavor-changing neutral currents

The quantities we are primarily interested in are **flavor-changing neutral currents (FCNCs)**. These are processes where a photon or a Z boson couple to a quark and change its flavor. Because A and Z are electrically neutral, they cannot change the electric charge of the quark, so this forbids transitions like $\{u, c, t\} \leftrightarrow \{d, s, b\}$ where the former carry charge $2/3$ and the latter $-1/3$. Still, FCNCs can induce transitions within each set $\{u, c, t\}$ or $\{d, s, b\}$.

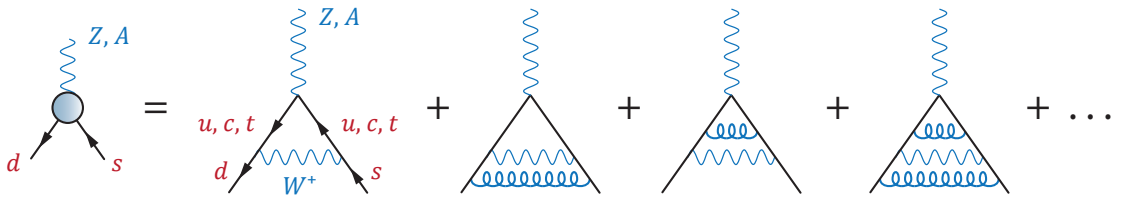
You see, however, from Fig. 1 that there is no Feynman rule describing such a process: When A, Z couple to a quark, they leave its flavor intact. This does not mean that such a process cannot happen in general: If, say, a strange quark emits a W^+ boson it can turn into an up, charm or top quark, which couples to A or Z and then reabsorbs the W^+ boson to turn into a different flavor like d. Or the strange quark could emit a W^+ boson that joins with another one into a three-boson vertex. Diagrammatically, these two options look like this:



(3)

So the full, *dressed* vertex between s and d quark containing all quantum loop effects does indeed exist, it just doesn't have a tree level contribution like in Eq. (1) and one needs at least a loop diagram. Because such loop diagrams are strongly suppressed by the large masses of the W bosons, the amplitude for this process will be very small. Additional suppression will also come from the weak coupling and the CKM matrix elements attached to the vertices. Therefore, such processes pose tight constraints on physics beyond the SM with possible new interactions that might allow them to happen already at tree level.

Now, there is an essential complication here because in order to compute such a vertex it is not sufficient to just calculate the one-loop diagrams above. Quarks are also subject to the strong interaction, so they can exchange arbitrarily many gluons:



(4)

Likewise, in the second diagram of Eq. (3) one could exchange arbitrarily many gluons between the quarks at the bottom. Gluons do not change the flavor, so every time a quark emits or absorbs a gluon its flavor stays intact. Unless the momenta are very large, the strong coupling is not small and higher-loop diagrams with massless gluons are not suppressed, so there is no point in calculating the vertex diagram by diagram – perturbation theory won't work. Instead, this calls for a nonperturbative treatment in QCD. A possible way to do so is to solve **Bethe-Salpeter equations (BSEs)**.

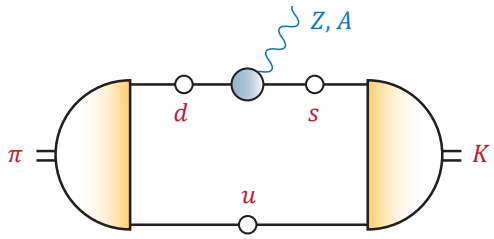


Figure 3: $K \to \pi$ transition matrix element

Another complication is that we cannot detect free quarks due to confinement. Thus, all quark lines must end in a hadron like in Fig. 2 and they can only appear internally inside a hadron. For the vertex discussed above, this means we ultimately need to study **transition matrix elements** like in Fig. 10, where for example a kaon ($u\bar{s}$) turns into a pion ($u\bar{d}$). But this is complicated by QCD effects: When we calculate the $K \to \pi$ transition, we also need to take into account all gluon effects which are nonperturbative. This is what makes practical calculations extremely hard, and nonperturbative QCD effects are usually the main obstacle in precision determinations of such flavor matrix elements.

3 Bethe-Salpeter equation

So how can we calculate the vertex in Eq. (4) nonperturbatively? A simple analogue is the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1\,. \tag{5}$$

Imagine that each term in the series stands for some Feynman diagram in QFT, which contains couplings that are small. Then for small $|x|$ you can stop the expansion after a few terms. On the other hand, summing up the series becomes impossible if the couplings become large, i.e. $|x| > 1$. But now imagine that someone *gave* you the equation

$$f(x) = 1 + x f(x) \quad \Leftrightarrow \quad f(x)^{-1} = 1 - x\,, \tag{6}$$

which has the solution $f(x) = 1/(1-x)$ for any x. If x is small, you can expand $f(x)$ in a perturbative series and stop after a few terms like before. However, if x becomes large, Eqs. (6) and (5) no longer agree: even though

$$f(x) = 1 + x f(x) = 1 + x + x^2 f(x) = 1 + x + x^2 + x^3 f(x) = \ldots \tag{7}$$

is exact at every step (simply reinsert the l.h.s. into the r.h.s.), in the geometric series we drop the last term which otherwise pulls the result back even if x becomes large, and this is what makes the perturbative expansion fail. In any case, this is not a serious problem because in that case we can just solve the non-perturbative equation directly.

Of course, this implies that we *know* the correct non-perturbative equation beforehand, but it turns out that quantum field theory gives us just that. The analogue of Eq. (6) for a quark-photon or quark-Z-boson vertex is the **Bethe-Salpeter equation (BSE)** shown in Fig. 4, which has the structural form

$$\Gamma = \Gamma_0 + K\, G_0\, \Gamma. \tag{8}$$

Γ is the full vertex, Γ_0 is the tree-level vertex given by the Feynman rules, K is the kernel that

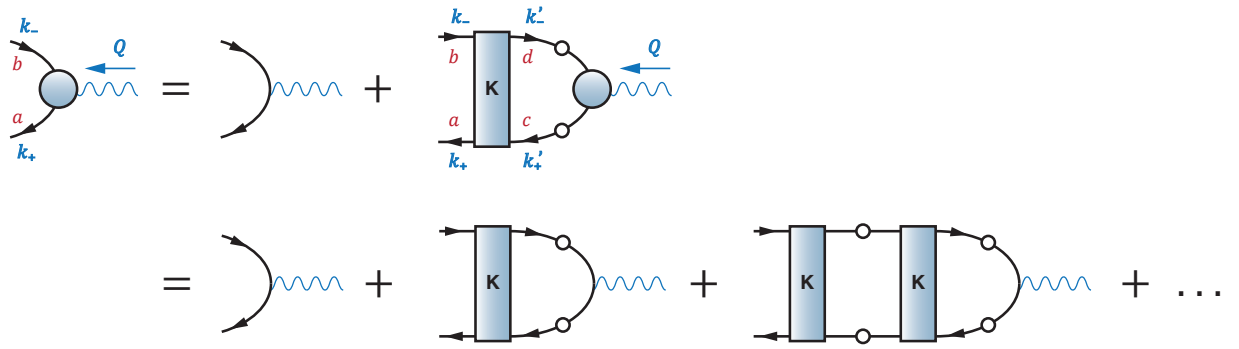


Figure 4: Inhomogeneous Bethe-Salpeter equation for the A/Z boson vertex

contains all possible interactions between quark and antiquark, G_0 is the product of two dressed quark propagators, which contain all possible self-interactions of the quarks, and each multiplication stands for a loop momentum integration. If you iterate this equation, you will generate all possible exchanges like those in Eq. (4):

$$\Gamma = \Gamma_0 + K\,G_0\,\Gamma_0 + K\,G_0\,K\,G_0\,\Gamma_0 + \ldots \tag{9}$$

So instead of summing up a (non-converging) series of *integrals*, we solve a nonperturbative *integral equation* for Γ. This is usually done by iteration: Start with some guess for Γ and iterate the equation until it converges. As long as $\Gamma_0 \neq 0$, the BSE will have a nontrivial solution because we can formally rewrite it as

$$\Gamma = (1 - K\,G_0)^{-1}\,\Gamma_0\,. \tag{10}$$

If the kernel is a one-gluon exchange, then the BSE solution automatically generates all possible gluon ladders *at once*, without the need for adding up diagrams. This is pretty efficient, right?

If we were just talking about QCD effects entering in the kernel K, Eq. (8) would indeed be the full equation. Of course, K is a priori unknown but there are ways to derive it systematically starting with a gluon exchange, crossed gluon exchanges, three-gluon vertices and so on. Note that repeated gluon exchanges cannot appear in the kernel because those are already generated by the iteration – the kernel K must be the 'two-particle irreducible' $q\bar{q}$ kernel.

Now, QCD does not change the flavor and neither do the A or Z couplings, so if an up quark comes in, an up quark must go out again. How can we then describe the vertex in Eq. (3), where the quark legs carry different flavor quantum numbers? Let's denote the general quark-photon or quark-Z-boson vertex by Γ_{ab}, where a quark with flavor b comes in and a quark with flavor a goes out (see Fig. 4). We observe the following:

- Because A and Z are neutral, the electric charges of the incoming and outgoing quarks must be the same, so vertices like Γ_{ud}, Γ_{du}, Γ_{us} etc. cannot exist in general. So again, a and b must have the same type $\{u, c, t\}$ or $\{d, s, b\}$.
- Only the flavor-diagonal vertices (Γ_{uu}, Γ_{dd}, Γ_{ss}, ...) have tree-level terms given by the Feynman rules. The flavor-nondiagonal vertices (Γ_{uc}, Γ_{cu}, Γ_{ds}, ...) do not have tree-level terms, so they can only be generated through loops with W bosons.
- If you start with Γ_{uu} and exchange a gluon, you still end up with two up quarks. If you exchange a W boson, the quarks on the other side of the exchange must have the opposite type (d, s or b), so they must end in a vertex Γ_{dd}, Γ_{ds}, Γ_{sd}, etc. This means we arrive at *coupled* equations.

In the following we drop the second diagram in Eq. (3), which is suppressed by two W propagators and their heavy masses, and restrict ourselves to the first diagram and hence exchanges of gluons and W bosons. Including two generations of quarks (up, down, strange, charm), we see that the structure of the equation must be as follows:

$$\left[\begin{array}{c} \Gamma_{uu} \\ \Gamma_{uc} \\ \Gamma_{cu} \\ \Gamma_{cc} \\ \hline \Gamma_{dd} \\ \Gamma_{ds} \\ \Gamma_{sd} \\ \Gamma_{ss} \end{array}\right] = \left[\begin{array}{c} \Gamma^0_{uu} \\ 0 \\ 0 \\ \Gamma^0_{cc} \\ \hline \Gamma^0_{dd} \\ 0 \\ 0 \\ \Gamma^0_{ss} \end{array}\right] + \left[\begin{array}{cccc|cccc} K & 0 & 0 & 0 & \mathcal{M}^{dd}_{uu} & \mathcal{M}^{ds}_{uu} & \mathcal{M}^{sd}_{uu} & \mathcal{M}^{ss}_{uu} \\ 0 & K & 0 & 0 & \mathcal{M}^{dd}_{uc} & \mathcal{M}^{ds}_{uc} & \mathcal{M}^{sd}_{uc} & \mathcal{M}^{ss}_{uc} \\ 0 & 0 & K & 0 & \mathcal{M}^{dd}_{cu} & \mathcal{M}^{ds}_{cu} & \mathcal{M}^{sd}_{cu} & \mathcal{M}^{ss}_{cu} \\ 0 & 0 & 0 & K & \mathcal{M}^{dd}_{cc} & \mathcal{M}^{ds}_{cc} & \mathcal{M}^{sd}_{cc} & \mathcal{M}^{ss}_{cc} \\ \hline \mathcal{M}^{uu}_{dd} & \mathcal{M}^{uc}_{dd} & \mathcal{M}^{cu}_{dd} & \mathcal{M}^{cc}_{dd} & K & 0 & 0 & 0 \\ \mathcal{M}^{uu}_{ds} & \mathcal{M}^{uc}_{ds} & \mathcal{M}^{cu}_{ds} & \mathcal{M}^{cc}_{ds} & 0 & K & 0 & 0 \\ \mathcal{M}^{uu}_{sd} & \mathcal{M}^{uc}_{sd} & \mathcal{M}^{cu}_{sd} & \mathcal{M}^{cc}_{sd} & 0 & 0 & K & 0 \\ \mathcal{M}^{uu}_{ss} & \mathcal{M}^{uc}_{ss} & \mathcal{M}^{cu}_{ss} & \mathcal{M}^{cc}_{ss} & 0 & 0 & 0 & K \end{array}\right] \left[\begin{array}{c} \Psi_{uu} \\ \Psi_{uc} \\ \Psi_{cu} \\ \Psi_{cc} \\ \hline \Psi_{dd} \\ \Psi_{ds} \\ \Psi_{sd} \\ \Psi_{ss} \end{array}\right] . \tag{11}$$

Here we abbreviated $\Psi_{ab} = G^0_{ab}\,\Gamma_{ab}$. $\mathcal{M}^{cd}_{ab}$ denotes a W-boson exchange between quarks ab on the left and cd on the right (see Fig. 4) and K is the QCD kernel which does not change the flavor. Thus, without W-boson exchanges the kernel becomes diagonal and the equations decouple to

$$\Gamma_{ab} = \delta_{ab}\,\Gamma^0_{ab} + K\,G^0_{ab}\,\Gamma_{ab}\,. \tag{12}$$

For $a = b$ we return to Eq. (8), where the vertices Γ_{uu}, Γ_{dd}, Γ_{ss}, Γ_{cc} differ only by the masses of the quarks entering in G^0_{aa} (i.e., $u\bar{u}$, $d\bar{d}$, $s\bar{s}$, $c\bar{c}$). For $a \neq b$, these become homogeneous equations which only admit the trivial solution $\Gamma_{ab} = 0$, i.e., they will iterate to zero: FCNCs cannot be constructed with QCD only, and it is the off-diagonal entries in the kernel that generate them. For the same reason we also cannot compute, say, Γ_{ds} alone as it would iterate to zero; we need at least one of the diagonal vertices with a term Γ^0_{aa} in the system.

So much for the general ideas, now let's get into the details. To work out Eq. (11), we need to establish a few things:

- In Sec. 4 we will have a closer look at the dressed quark propagator that goes into the BSE, and how to calculate it self-consistently from its own (Dyson-Schwinger) equation.
- In Sec. 5 we will work out the BSE for the quark-photon vertex, which describes the coupling to a photon and with QCD effects only.

The code you got already takes care of these points, i.e., it computes the quark propagator and the quark-photon vertex. This was part of last year's internship project, so I simply copied over the respective descriptions. You will need some elements of these sections to understand what the code actually does, but for the present project you can jump directly to the next point:

- In Sec. 6 we will include the weak interactions and see what needs to be generalized to arrive at the coupled system of equations (11) for the quark-Z-boson vertex.

Two more remarks:

- In the following we will use a Euclidean metric instead of a Minkowski metric, so the formulas might look slightly different from what you find in textbooks. More details on Euclidean conventions can be found in Appendix B.
- Just in case you're not overly familiar with gamma matrices yet, don't worry – in the end all Dirac traces are worked out explicitly, so all you need to code are scalar equations.

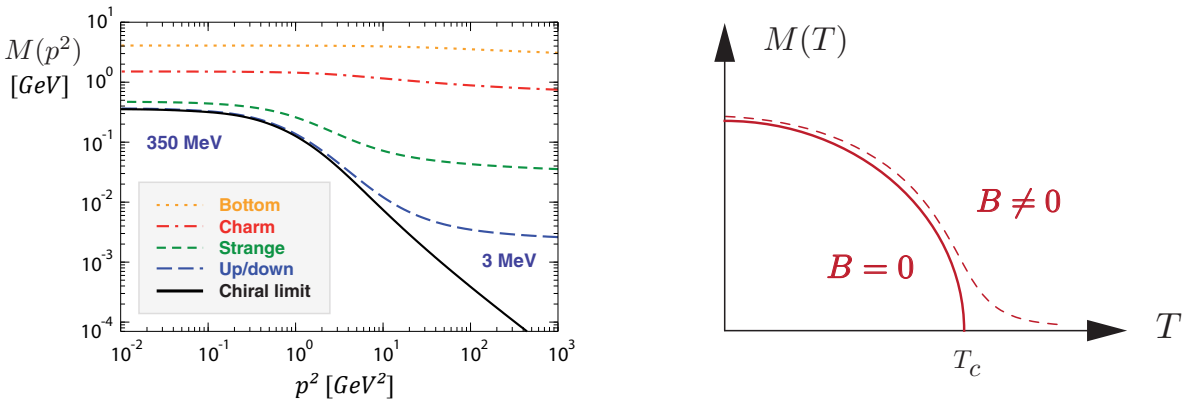


Figure 5: Quark mass function for different flavors [1]; magnetization as a function of temperature

4 Quark Propagator

The basic quantity that describes a particle in a quantum field theory (QFT) is its **propagator**. It tells us how a particle 'propagates' from one point in spacetime to another, or after a Fourier transform, how its modes are distributed in momentum space. For example, the propagator of a free particle has the form

$$\frac{1}{p^2 + m^2}\,, \tag{13}$$

where m is the mass of the particle and p^2 is the squared momentum (which is a Lorentz-invariant variable). This doesn't tell us much except for the fact that it has a *pole* for some negative value of $p^2 = -m^2$, which is the free particle pole. Very roughly speaking, this is how a QFT allows us to extract the mass of some particle or intermediate resonance: When you see a bump or a peak in an experimental cross section, then this corresponds to a pole in some propagator or some scattering amplitude, and the momentum or energy where the peak appears defines the mass of that particle. These poles do not necessarily sit on the real negative p^2 axis but they can also appear in the complex plane $p^2 \in \mathbb{C}$; in fact, the resonance peaks we observe in experiments correspond to poles in the complex plane on higher Riemann sheets.

Concerning quarks, the story becomes more interesting for several reasons. First of all, quarks are spin-1/2 particles which means that the most general possible form of the quark propagator $S(p)$ according to Lorentz invariance is

$$S(p)^{-1} = A(p^2)\left(i\not{p} + M(p^2)\right) \quad \Leftrightarrow \quad S(p) = \frac{1}{A(p^2)}\,\frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2} = -i\not{p}\,\sigma_v(p^2) + \sigma_s(p^2)\,. \tag{14}$$

(See Sec. B for the definition of gamma matrices and slashes.) It is expressed in terms of two dressing functions, which can be chosen either as $A(p^2)$ and the **quark mass function** $M(p^2)$, or equivalently as the two dressing functions $\sigma_v(p^2)$ and $\sigma_s(p^2)$ — in any case, two of them are needed to describe the propagator of a quark. The variable p^2 can take any value $p^2 \in \mathbb{C}$, although in the following we are mainly interested in real and positive momenta, $p^2 \in \mathbb{R}_+$. For a free spin-1/2 particle, the formulas above simplify to $A(p^2) = 1$ and $M(p^2) = m$, where m is the mass of the particle; in that case, the propagator becomes

$$S_0(p) = \frac{-i\not{p} + m}{p^2 + m^2} \quad \Leftrightarrow \quad \sigma_v(p^2) = \frac{1}{p^2 + m^2}\,, \quad \sigma_s(p^2) = \frac{m}{p^2 + m^2}\,. \tag{15}$$

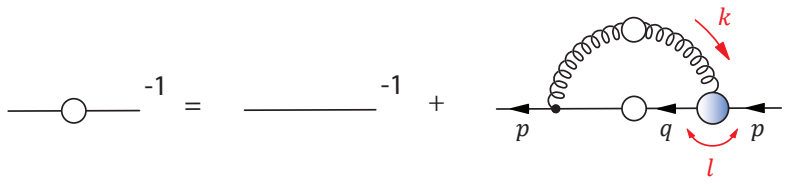


Figure 6: Momentum routing in the quark DSE.

For interacting particles we should not expect to find a structure like in Eq. (15), i.e., $A(p^2) \neq 1$ and $M(p^2) \neq m$, although in practice they can look quite similar. What makes quarks rather special is that they are **confined** inside hadrons, so their analytic structure may look drastically different. In principle the functions $\sigma_v(p^2)$ and $\sigma_s(p^2)$ can have (many) poles or branch cuts in the complex plane of p^2, and their analytic structure can tell us something about confinement: For example, a sufficient confinement criterion is if the quark propagator does *not* have poles on the real negative p^2 axis but in the complex plane, which prevents a free particle interpretation. However, the actual analytic structure of the quark propagator in QCD is still unknown.

Light quarks are also special for another reason, because most of their mass is dynamically generated in QCD through their interactions with gluons. You can see this from the fact that light u and d quarks have masses of about 3–5 MeV; these are the 'current-quark' masses in the QCD Lagrangian that arise from the Higgs mechanism and are an external input to QCD. However, the mass of a proton (three quarks) is 940 MeV, which means that 98–99% of the mass of the proton (and therefore nuclei and atoms) must be somehow produced in QCD. This feature can be understood through **spontaneous chiral symmetry breaking**, which is a non-perturbative effect that cannot be directly seen from the QCD Lagrangian. It emerges through the dynamics of quarks and gluons, i.e., their interactions cause the quarks to gain mass, and this is encoded in the mass function $M(p^2)$. At large momenta, $M(p^2)$ becomes the current-quark mass m, whereas at small momenta it is much larger (about 300–400 MeV) and thereby defines a 'constituent-quark mass' (Fig. 5), whichs is the relevant mass scale for the proton and other hadrons.

An analogy for this is the magnetization of a magnet as a function of the temperature, where a magnetization spontaneously occurs below a critical temperature. If an external magnetic field B is switched on, the magnetization persists also for large temperatures. In QCD, the role of T is played by the momentum and the analogue of B is the current-quark mass in the Lagrangian.

The fact that this effect is non-perturbative means that we cannot produce it at any order of **perturbation theory**. Perturbation theory is one of the main tools to make a QFT useful in practice; it means that one expands some scattering amplitude around a small coupling parameter and sums up Feynman diagrams: all one-loop diagrams with all possible intermediate particles, all two-loop diagrams, etc., and if the couplings are small enough one can stop the series after a few terms. Perturbation theory works extremely well for the electromagnetic interaction described by QED since the electromagnetic coupling $\alpha_{\text{QED}} \approx 1/137$ is indeed very small. For QCD it only works well as long as we are interested only in very large momenta, where also the strong coupling α_{QCD} becomes small; this is what allows us to describe high-energy scattering processes in QCD. However, for small momenta α_{QCD} becomes large and perturbation theory no longer works.

Instead, one can derive the **Dyson-Schwinger equations (DSEs)** from the Lagrangian, which are the quantum equations of motion of a QFT. They are exact and therefore non-perturbative. The quark DSE is shown in Fig. 6 and determines the quark propagator $S(p)$:

$$S(p)^{-1} = Z_2 \left(i\not{p} + m_0\right) + \Sigma(p)\,, \tag{16}$$

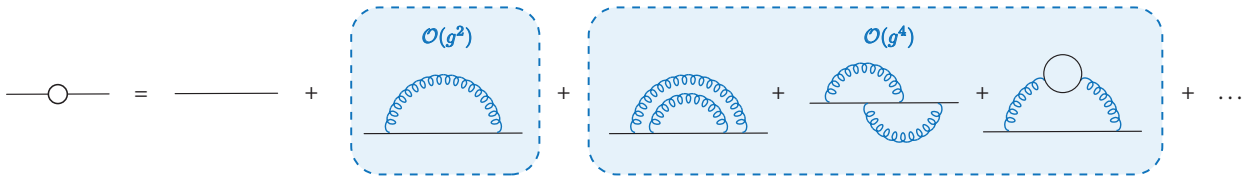


Figure 7: Perturbative expansion of the quark propagator

where $\Sigma(p)$ is the quark self-energy that contains the gluon propagator, the quark-gluon vertex and again the quark propagator. m_0 is the bare current quark mass that enters in the QCD Lagrangian, and Z_2 is the quark renormalization constant. Observe that the equation has the structural form of Eq. (6), $f(x)^{-1} = 1 - x$ (the minus is absorbed in the self-energy). Thus, if the QCD coupling contained in $\Sigma(p)$ becomes small, we can expand $S(p)$ into a series like in Eq. (7) by reinserting the equation at every instance where the quark propagator appears inside the loop, and stop after a few terms — this is the perturbative series for the quark propagator shown in Fig. 7. However, if the coupling becomes large we have no choice but to solve the equation directly.

The self-energy has the explicit form

$$\Sigma(p) = -\frac{4g^2}{3} Z_\Gamma \int_q i\gamma^\mu \, S(q) \, D^{\mu\nu}(k) \, \Gamma^\nu(l,k) \tag{17}$$

and contains the following ingredients:

- g is the strong coupling ($\alpha_{\rm QCD} = g^2/(4\pi)$) and the prefactor $4/3$ comes from the color trace.
- The dressed quark propagator $S(q)$ appears again inside the loop, so the DSE is an integral equation which determines $S(p)$.
- The dressed gluon propagator $D^{\mu\nu}(k)$ depends on the gluon momentum $k = q - p$. We will work in Landau gauge, where it is given by

$$D^{\mu\nu}(k) = \frac{Z(k^2)}{k^2} T_k^{\mu\nu}, \qquad T_k^{\mu\nu} = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \tag{18}$$

  where $Z(k^2)$ is the gluon dressing function and $T_k^{\mu\nu}$ is a transverse projector.
- The bare quark-gluon vertex $g\, Z_\Gamma \, i\gamma^\mu$ comes with a renormalization constant Z_Γ.
- The dressed quark-gluon vertex $g\,\Gamma^\mu(l,k)$ depends on the average quark momentum $l = (q+p)/2$ and the gluon momentum k. In principle it consists of 12 Lorentz-Dirac tensors, but we restrict ourselves to the **rainbow-ladder truncation** which is defined by the ansatz

$$\Gamma^\mu(l,k) = f(k^2)\, i\gamma^\mu \,, \tag{19}$$

  where the dressing function $f(k^2)$ depends on the gluon momentum only.
- The integral measure $\int_q$ is given by

$$\int_q = \int \frac{d^4q}{(2\pi)^4} = \frac{1}{(2\pi)^4} \frac{1}{2} \int_0^{L^2} dq^2 \, q^2 \int_{-1}^{1} dz \, \sqrt{1-z^2} \int_{-1}^{1} dy \int_0^{2\pi} d\phi \,, \tag{20}$$

  where L is the cutoff in the system (a typical value is $L = 10^3$ GeV) and we use hyperspherical

variables:

$$p^\mu = \sqrt{p^2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \qquad q^\mu = \sqrt{q^2}\begin{bmatrix}\sqrt{1-z^2}\,\sqrt{1-y^2}\,\sin\phi\\ \sqrt{1-z^2}\,\sqrt{1-y^2}\,\cos\phi\\ \sqrt{1-z^2}\,y\\ z\end{bmatrix}. \tag{21}$$

Because in the end we will break down the quark DSE into two Lorentz-invariant equations for $A(p^2)$ and $M(p^2)$ and the only Lorentz invariants in the system are p^2, q^2 and $p\cdot q = \sqrt{p^2}\sqrt{q^2}\,z$, the two integrations over the variables y and ϕ become trivial so that only the two integrations over q^2 and z remain. Therefore, we can equivalently work with the vector

$$q^\mu = \sqrt{q^2}\begin{bmatrix}0\\0\\\sqrt{1-z^2}\\z\end{bmatrix}. \tag{22}$$

(More on Euclidean conventions and four-vectors can be found in Appendix B.)

Putting things together, we arrive at

$$\Sigma(p) = \frac{16\pi}{3}\,Z_2^2\int\limits_q \frac{\alpha(k^2)}{k^2}\,T_k^{\mu\nu}\gamma^\mu S(q)\gamma^\nu\,, \qquad \alpha(k^2) = \frac{g^2}{4\pi}\,\frac{Z_\Gamma}{Z_2^2}\,Z(k^2)\,f(k^2)\,, \tag{23}$$

where the **effective interaction** $\alpha(k^2)$ absorbs the dressing functions of the gluon propagator and quark-gluon vertex. Because we do not know these quantities (we would need to solve their own DSEs for that), we employ the following ansatz for $\alpha(k^2)$ to solve the quark DSE:

$$\alpha(k^2) = \pi\eta^7 x^2\,e^{-\eta^2 x} + \frac{2\pi\gamma_m\left(1-e^{-k^2/\Lambda_t^2}\right)}{\ln\left[e^2-1+\left(1+k^2/\Lambda_{\rm QCD}^2\right)^2\right]}\,, \qquad x = \frac{k^2}{\Lambda^2}\,, \tag{24}$$

which is the so-called Maris-Tandy model [2, 3]. The second term with the parameters $\Lambda_t = 1$ GeV, $\Lambda_{\rm QCD} = 0.234$ GeV and $\gamma_m = 12/25$ is only relevant for large momenta, where it ensures the correct perturbative behavior but is otherwise not essential. By contrast, the first term with the parameters $\Lambda = 0.72$ GeV and $1.6 \lesssim \eta \lesssim 2$ dominates the small-momentum behavior and is important for the dynamical generation of a quark mass (in practice, you can use $\eta = 1.85$).

Solving the quark DSE in practice (more technical)

To work out the explicit form of the quark DSE, we can make it more compact by abbreviating

$$g(k^2) = Z_2^2\,\frac{16\pi}{3}\,\frac{\alpha(k^2)}{k^2} \quad\Rightarrow\quad \Sigma(p) = \int\limits_q g(k^2)\,T_k^{\mu\nu}\gamma^\mu S(q)\gamma^\nu\,. \tag{25}$$

Like Eq. (14) for the quark propagator, the most general form of the self-energy determined by Lorentz covariance is

$$\Sigma(p) = i\not{p}\,\Sigma_A(p^2) + \Sigma_M(p^2)\,. \tag{26}$$

If we plug this into the quark DSE and compare the coefficients of the Dirac matrices, we arrive at two coupled integral equations for the dressing functions $A(p^2)$ and $M(p^2)$:

$$\begin{aligned} A(p^2) &= Z_2 + \Sigma_A(p^2)\,,\\ M(p^2)A(p^2) &= Z_2\,m_0 + \Sigma_M(p^2)\,. \end{aligned} \tag{27}$$

It turns out that the self-energy integrals would not converge if we sent the cutoff L in the integrals to infinity; they are logarithmically divergent in the ultraviolet (UV). This is a general feature of QFTs and has a deeper underlying meaning, but in practice it just tells us that we must employ **renormalization conditions**, i.e., we demand that

$$A(\mu^2) \stackrel{!}{=} 1\,, \qquad M(\mu^2) \stackrel{!}{=} m\,, \tag{28}$$

where m is the renormalized current-quark mass at some arbitrary renormalization point $p^2 = \mu^2$ (this is the current-quark mass we should compare to experiment). From Eq. (30), this yields

$$Z_2 = 1 - \Sigma_A(\mu^2)\,, \qquad m_0 = \frac{m - \Sigma_M(\mu^2)}{1 - \Sigma_A(\mu^2)}\,, \tag{29}$$

and plugging this back into Eq. (30) yields the final form of the DSEs:

$$\begin{aligned} A(p^2) &= 1 + \Sigma_A(p^2) - \Sigma_A(\mu^2)\,, \\ M(p^2)A(p^2) &= m + \Sigma_M(p^2) - \Sigma_M(\mu^2)\,. \end{aligned} \tag{30}$$

Through the subtraction, the divergences cancel and $A(p^2)$ and $M(p^2)$ are finite.

To obtain $\Sigma_A(p^2)$ and $\Sigma_M(p^2)$, we take Dirac traces (recall that $\mathrm{Tr}\,\not{p} = 0$ and $\not{p}^2 = p^2$):

$$\frac{1}{4}\,\mathrm{Tr}\,\Sigma(p) = \Sigma_M(p^2)\,, \qquad \frac{1}{4p^2}\,\mathrm{Tr}\left\{-i\not{p}\,\Sigma(p)\right\} = \Sigma_A(p^2)\,. \tag{31}$$

Applying this to Eq. (25) yields the expressions

$$\Sigma_A(p^2) = \int_q \sigma_v(q^2)\, g(k^2)\, F(p^2, q^2, z), \qquad \Sigma_M(p^2) = 3\int_q \sigma_s(q^2)\, g(k^2) \tag{32}$$

which involve the quark dressings $\sigma_v(q^2)$, $\sigma_s(q^2)$ that can be reconstructed from $A(q^2)$ and $M(q^2)$. The squared gluon momentum is $k^2 = p^2 + q^2 - 2\,p\cdot q = p^2 + q^2 - 2pqz$ (we now abbreviate $p = \sqrt{p^2}$ and $q = \sqrt{q^2}$), and the dimensionless quantity F is given by

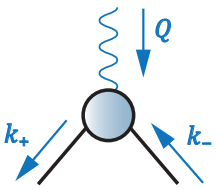
$$\begin{aligned} p^2\,F(p^2, q^2, z) &= -\tfrac{1}{4}\,\mathrm{Tr}\left\{\not{p}\,\gamma^\mu\,\not{q}\,\gamma^\nu\right\} T_k^{\mu\nu} = p\cdot q + \frac{2}{k^2}\,(p\cdot k)(q\cdot k) \\ &= 3\,p\cdot q - \frac{2}{k^2}\left(p^2\,q^2 - (p\cdot q)^2\right) = 3pqz - \frac{2p^2q^2}{k^2}\,(1 - z^2) \\ &= -k^2 + \frac{p^2 + q^2}{2} + \frac{(p^2 - q^2)^2}{2k^2} = p^2 + 3\,p\cdot k + 2\,\frac{(p\cdot k)^2}{k^2}\,. \end{aligned} \tag{33}$$

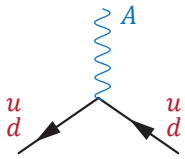
Either of these forms are equally good to calculate the self-energy integrals, e.g.

$$F(p^2, q^2, z) = \frac{3qz}{p} - \frac{2q^2}{k^2}\,(1 - z^2)\,. \tag{34}$$

Eqs. (30) can be solved iteratively. Start with some guess for $A(p^2)$ and $M(p^2)$ (e.g., set them to 1) and calculate $\Sigma_A(p^2)$ and $\Sigma_M(p^2)$. From there, compute the new functions $A(p^2)$, $M(p^2)$ according to Eq. (30) and determine the renormalization constant Z_2 from Eq. (29), which enters again in the self-energy in the next step. Repeat the procedure until it converges. For a spacelike (i.e., positive and real) external momentum $p^2 \in \mathbb{R}_+$, the squared gluon momentum k^2 is also real and positive and the coupled system can be solved without complications.

The renormalized current-quark mass m is an input, and because we do not distinguish between up and down quarks a typical value for both is $m_u = m_d = 4$ MeV. These are the renormalized values at the renormalization point $\mu = 19$ GeV, and they are chosen to reproduce the pion mass obtained with the Maris-Tandy interaction. Likewise, for strange quarks the respective current quark mass is $m_s \sim 90$ MeV and for charm quarks it is $m_c \sim 800$ MeV; these are fixed by the masses of the D and D_s mesons [4].





$$eQ_f Z_2\, i\gamma^\mu\,,$$

Figure 8: Momentum routing in the quark-photon vertex (left) and Feynman rule for the tree-level vertex (right)

5 Quark-photon vertex

A well-established and time-honored way to probe the internal structure of hadrons is to measure their **electromagnetic form factors**. For example, the leading diagram in electron-nucleon scattering is a one-photon exchange, where a virtual photon couples to the nucleon. In general, the coupling of a photon to an onshell spin-1/2 fermion with mass m is described by the electromagnetic current matrix element

$$J^\mu(k,Q) = i\bar{u}(k_+)\left[F_1(Q^2)\,\gamma^\mu - \frac{F_2(Q^2)}{2m}\,\sigma^{\mu\nu}Q^\nu\right]u(k_-)\,. \tag{35}$$

Here, Q^μ is the photon four-momentum, $k_\pm^\mu$ are the outgoing and incoming momenta of the fermion, $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu,\gamma^\nu]$, and $u(k_\pm)$ are the Dirac spinors satisfying the Dirac equation:

$$\not{p}\,u(p) = imu(p)\,, \qquad \bar{u}(p)\not{p} = im\bar{u}(p) \qquad \text{with} \quad p^2 = -m^2\,. \tag{36}$$

Because the fermion is onshell, we have $k_\pm^2 = -m^2$. If we define the average momentum k by

$$k_\pm = k \pm \frac{Q}{2} \quad \Rightarrow \quad k_\pm^2 = k^2 + \frac{Q^2}{4} \pm k\cdot Q = -m^2\,, \tag{37}$$

then it follows that $k^2 = -m^2 - Q^2/4$ and $k\cdot Q = 0$, so that the process is fully characterized by the squared momentum transfer $Q^2 \geq 0$. The electromagnetic structure of the fermion is thus described by its Lorentz-invariant form factors, the **Dirac** form factor $F_1(Q^2)$ and the **Pauli** form factor $F_2(Q^2)$. For vanishing photon momentum transfer $Q^2 = 0$, the Pauli form factor $F_2(0)$ defines the **anomalous magnetic moment** of the fermion.

When a photon couples to a hadron, then microscopically it must always couple to a quark. The elementary quantity is therefore the dressed **quark-photon vertex**, which describes the electromagnetic coupling of quarks to photons and thus encodes their electromagnetic properties. An elementary quark is, however, not *onshell* and does not have a well-defined mass. This means we must relax the constraints above, i.e., we neither contract with onshell spinors on the left and right nor do we impose $k_\pm^2 = -m^2$. As a result, the general quark-photon vertex depends on three Lorentz invariants k^2, Q^2 and $k\cdot Q$, and it features a much richer Lorentz-Dirac tensor structure: Instead of γ^μ and $\sigma^{\mu\nu}Q^\nu$ that enter in Eq. (35) it depends on 12 tensors, and instead of two form factors $F_1(Q^2)$ and $F_2(Q^2)$ it will thus depend on 12 dressing functions $F_i(k^2, k\cdot Q, Q^2)$ that describe the electromagnetic properties of the quark.

The kinematics in the quark-photon vertex $\Gamma^\mu(k,Q)$ are shown in Fig. 8: The vertex depends on two independent momenta, the incoming quark momentum k_- and outgoing momentum k_+ (these are four-vectors), or equivalently the photon momentum Q and relative quark momentum k

which are related by

$$k_\pm = k \pm \frac{Q}{2} \quad \Leftrightarrow \quad Q = k_+ - k_- \,, \quad k = \frac{k_+ + k_-}{2} \quad \Rightarrow \quad k_\pm^2 = k^2 + \frac{Q^2}{4} \pm k \cdot Q \,. \tag{38}$$

The tree-level vertex is given by the Feynman rule in Fig. 8, where e is the unit of electric charge (in natural units) such that $\alpha_{QED} = e^2/(4\pi) \approx 1/137$ is the electromagnetic coupling, $Q_f = 2/3$ or $-1/3$ is the electric charge of the respective quark, and Z_2 is the quark renormalization constant from the previous section. In the following we will split off the overall factor eQ_f and consider the quark-photon vertex for one flavor only, whose tree-level expression is $Z_2\, i\gamma^\mu$.

The full quark-photon vertex depends on 12 linearly independent Lorentz-Dirac tensors. There is no common convention for them in the literature, but it is convenient to work with the following decomposition [1]:

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{4} g_j(k^2,\omega,Q^2)\, iG_j^\mu(k,Q) + \sum_{j=1}^{8} f_j(k^2,\omega,Q^2)\, iT_j^\mu(k,Q) \,. \tag{39}$$

Here, the $g_j(k^2,\omega,Q^2)$ and $f_j(k^2,\omega,Q^2)$ are the 12 dressing functions which depend on the three possible Lorentz invariants k^2, $\omega = k\cdot Q$ and Q^2, and the G_j^μ and T_j^μ are the corresponding tensors:

$$\begin{aligned}
G_1^\mu &= \gamma^\mu \,, & T_1^\mu &= t_{QQ}^{\mu\nu}\,\gamma^\nu \,, & T_5^\mu &= t_{QQ}^{\mu\nu}\, ik^\nu \,, \\
G_2^\mu &= k^\mu \not{k} \,, & T_2^\mu &= \omega\, t_{QQ}^{\mu\nu}\, \tfrac{i}{2}[\gamma^\nu, \not{k}] \,, & T_6^\mu &= t_{QQ}^{\mu\nu}\, k^\nu \not{k} \,, \\
G_3^\mu &= ik^\mu \,, & T_3^\mu &= \tfrac{i}{2}[\gamma^\mu, \not{Q}] \,, & T_7^\mu &= \omega\, t_{Qk}^{\mu\nu}\, \gamma^\nu \,, \\
G_4^\mu &= \omega\, \tfrac{i}{2}[\gamma^\mu, \not{k}] \,, & T_4^\mu &= \tfrac{1}{6}[\gamma^\mu, \not{k}, \not{Q}] \,, & T_8^\mu &= t_{Qk}^{\mu\nu}\, \tfrac{i}{2}[\gamma^\nu, \not{k}] \,.
\end{aligned} \tag{40}$$

The quantities $t_{ab}^{\mu\nu}$ are defined by

$$t_{ab}^{\mu\nu} = a\cdot b\, \delta^{\mu\nu} - b^\mu a^\nu \,, \tag{41}$$

which is convenient because it entails that

$$a^\mu\, t_{ab}^{\mu\nu} = a\cdot b\, a^\nu - a\cdot b\, a^\nu = 0 \,, \qquad t_{ab}^{\mu\nu}\, b^\nu = a\cdot b\, b^\mu - a\cdot b\, b^\mu = 0 \,. \tag{42}$$

The commutators are given by

$$[A,B] = AB - BA \,, \qquad [A,B,C] = [A,B]\,C + [B,C]\,A + [C,A]\,B \,. \tag{43}$$

The decomposition (39) is advantageous for several reasons:

- The quark-photon vertex must satisfy electromagnetic gauge invariance in the form of the so-called Ward-Takahashi identity (WTI)

  $$Q^\mu \Gamma^\mu(k,Q) = S(k_+)^{-1} - S(k_-)^{-1} \,, \tag{44}$$

  which relates its longitudinal part with the quark propagator. From Eqs. (42–43) you can see that the tensors T_j^μ are transverse to the photon momentum Q^μ, i.e., $Q^\mu T_j^\mu = 0$, so they drop out from the WTI. Therefore, the WTI only affects the dressing functions g_j which are completely determined by the quark propagator. Inserting the decomposition (14), one can easily show that this entails

  $$g_1 = \Sigma_A \,, \qquad g_2 = 2\Delta_A \,, \qquad g_3 = -2\Delta_B \,, \qquad g_4 = 0 \,, \tag{45}$$

  where

  $$\Sigma_A = \frac{A(k_+^2) + A(k_-^2)}{2} \,, \qquad \Delta_A = \frac{A(k_+^2) - A(k_-^2)}{k_+^2 - k_-^2} \tag{46}$$

  and likewise for Δ_B, where $B(p^2) = A(p^2)\, M(p^2)$ and $k_+^2 - k_-^2 = 2k\cdot Q = 2\omega$. Therefore, once we know the two dressing functions $A(p^2)$ and $M(p^2)$ of the quark propagator, we already know a great deal about the quark-

photon vertex! Note in particular that for a tree-level propagator ($A = 1$, $M = m$) Eq. (45) reduces to $g_1 = 1$ and all other $g_j = 0$, so the vertex simplifies to

$$\Gamma^\mu(k,Q) = i\gamma^\mu + [\text{transverse part}]. \tag{47}$$

- The vertex has a charge-conjugation symmetry and must satisfy

$$\bar{\Gamma}^\mu(k,Q) := C\,\Gamma^\mu(-k,-Q)^T\,C^T \stackrel{!}{=} -\Gamma^\mu(k,-Q)\,, \tag{48}$$

where $C = \gamma^4\gamma^2$ is the charge-conjugation matrix and the superscript T denotes a Dirac matrix transpose. One can show that each individual basis element G_j^μ, T_j^μ satisfies the same relation, i.e., we already chose the basis to satisfy Eq. (48). As a consequence, the dressing functions g_j and f_j must be even in the variable ω, i.e., they can only depend on ω^2. This is easily verified for the g_j in Eq. (45): If we perform a Taylor expansion with respect to the variable $k_\pm^2$ from Eq. (38), we find

$$A(k_\pm^2) = A\left(k^2 + \frac{Q^2}{4} \pm \omega\right) = A\left(k^2 + \frac{Q^2}{4}\right) \pm \omega A'\left(k^2 + \frac{Q^2}{4}\right) + \ldots, \tag{49}$$

$$\Rightarrow \quad \Sigma_A = A\left(k^2 + \frac{Q^2}{4}\right) + \mathcal{O}(\omega^2)\,, \qquad \Delta_A = A'\left(k^2 + \frac{Q^2}{4}\right) + \mathcal{O}(\omega^2)\,, \tag{50}$$

where the primes denote the derivatives with respect to the arguments.

- The tensor basis (40) is 'free of kinematic constraints', which entails that all dressing functions $g_j(k^2,\omega,Q^2)$ and $f_j(k^2,\omega,Q^2)$ become constant in either of the kinematic limits $Q^\mu = 0$ or $k^\mu = 0$. Again, this is easily verified for the g_j since we have

$$\begin{aligned} Q^2 = 0: &\quad \Sigma_A = A(k^2)\,, \quad \Delta_A = A'(k^2)\,, \\ k^2 = 0: &\quad \Sigma_A = A\left(\frac{Q^2}{4}\right), \quad \Delta_A = A'\left(\frac{Q^2}{4}\right), \end{aligned} \tag{51}$$

All in all, even without knowing anything about the dynamics of the system, in this way we have derived some very useful constraints on its dressing functions from symmetry arguments alone.

To calculate the quark-photon vertex dynamically, we must solve its **Bethe-Salpeter equation** (BSE) shown in Fig. 4. This is again an exact, nonperturbative equation in QCD. However, it requires knowledge of the quark-antiquark kernel K, which contains the sum of all possible $q\bar{q}$ irreducible gluon interactions between the quarks that are mediated by the strong coupling α_{QCD}. (In principle it also contains the electromagnetic interactions through photon exchanges, but those are much smaller since they involve α_{QED}.)

In practice we will use again an ansatz for the kernel. However, this ansatz cannot be completely arbitrary because the kernel is closely linked with the quark propagator. In particular, the aforementioned property of electromagnetic gauge invariance can only hold if the kernel of the BSE is consistent with that of the quark DSE. Since we already settled on a rainbow-ladder truncation for the quark DSE, we must do the same for the vertex BSE. Therefore, we approximate the full kernel by a **gluon exchange** between the quark and antiquark, where we use the same effective interaction from Eqs. (24) and (25). The resulting BSE reads explicitly:

$$\Gamma^\mu(k,Q) = Z_2\, i\gamma^\mu - \int_{k'} g(l^2)\, T_l^{\alpha\beta}\, \gamma^\alpha\, S(k'_+)\, \Gamma^\mu(k',Q)\, S(k'_-)\, \gamma^\beta\,. \tag{52}$$

Here, k' is the relative momentum of the vertex inside the momentum loop, $k'_\pm = k' \pm Q/2$ are the quark momenta, $l = k - k'$ is the gluon momentum, and $T_l^{\mu\nu}$ is the transverse projector defined in Eq. (18). Keep in mind that $\Gamma^\mu(k,Q)$ is a 4×4 Dirac matrix with another Lorentz index $\mu = 1\ldots4$.

When you insert the tensor decomposition (39) on both sides of the equation, you can see that it eventually becomes a coupled system of integral equations for the 12 dressing functions g_j and f_j. Since we already *know* the g_j from the quark propagator, Eq. (45), what the BSE really determines are the eight transverse dressing functions f_j which contain the dynamics of the vertex.

Solving the quark-photon vertex BSE in practice (technical)

In principle, Eqs. (24) and (52) are all we need to solve the vertex BSE numerically. Suppose we write the decomposition (39) as

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{12} F_j(k^2,\omega,Q^2)\, i t_j^\mu(k,Q)\,, \tag{53}$$

with 12 dressing functions $F_j \in \{g_j, f_j\}$ and corresponding tensors $t_j^\mu \in \{G_j^\mu, T_j^\mu\}$, and plug this into the BSE. To project out the F_j on the l.h.s., we contract the equation with the charge-conjugate basis elements $\bar{t}_i^\mu$, with charge conjugation defined as in Eq. (48). The t_i^μ are not orthonormal, so we have

$$H_{ij}(k^2,\omega,Q^2) = \frac{1}{4}\,\mathrm{Tr}\,\left\{\bar{t}_i^\mu(k,Q)\, t_j^\mu(k,Q)\right\} \neq \delta_{ij}\,. \tag{54}$$

As a result, the BSE turns into

$$H_{ij}(k^2,\omega,Q^2)\, F_j(k^2,\omega,Q^2) = F_i^0(k^2,\omega,Q^2) + \int_{k'} K_{ij}\, F_j(k'^2,\omega',Q^2) \tag{55}$$

with

$$\begin{aligned} F_j^0(k^2,\omega,Q^2) &= Z_2\,\frac{1}{4}\,\mathrm{Tr}\,\left\{\bar{t}_i^\mu(k,Q)\,\gamma^\mu\right\}\,, \\ K_{ij} &= g(l^2)\, T_l^{\alpha\beta}\,\frac{1}{4}\,\mathrm{Tr}\,\left\{\bar{t}_i^\mu(k,Q)\,\gamma^\alpha\, S(k'_+)\, t_j^\mu(k',Q)\, S(k'_-)\,\gamma^\beta\right\}\,. \end{aligned} \tag{56}$$

The matrices H and K are known, so this is an inhomogeneous linear integral equation for the F_i. Note that it involves a matrix inversion for H.

There is still a more efficient way to handle the problem. The underlying idea [5] is to construct an **orthonormal basis** where $H_{ij} = \delta_{ij}$, so that we can bypass the matrix inversion. In addition, we will see that the resulting equation forms two orthogonal subspaces which can be handled independently, and the traces become very simple so we can work them out directly. To start with, we write down the four-vectors in a given coordinate frame:

$$Q^\mu = \sqrt{Q^2}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \qquad k^\mu = \sqrt{k^2}\begin{bmatrix}0\\0\\\sqrt{1-z^2}\\z\end{bmatrix}, \qquad k'^\mu = \sqrt{k'^2}\begin{bmatrix}0\\\sqrt{1-z'^2}\,\sqrt{1-y^2}\\\sqrt{1-z'^2}\,y\\z'\end{bmatrix}. \tag{57}$$

How exactly we distribute the components inside the four-vectors is irrelevant because all that matters in the end are the six possible Lorentz invariants Q^2, k^2, k'^2, $k\cdot Q$, $k'\cdot Q$ and $k\cdot k'$, which determine the system completely and correspond to the six entries in the vectors, where the three angular variables are related to z, z', $y \in [-1,1]$. However, in this way we can define alternative vectors

$$d^\mu = \hat{Q}^\mu = \frac{Q^\mu}{\sqrt{Q^2}}, \qquad r^\mu = \widehat{k_\perp}^\mu = \frac{k_\perp^\mu}{\sqrt{k_\perp^2}}\,, \qquad k_\perp^\mu = k^\mu - \frac{k\cdot Q}{Q^2}\,Q^\mu\,. \tag{58}$$

If we perform these operations on the four-vectors in Eq. (57), we arrive at

$$d^\mu = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \qquad r^\mu = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \qquad r'^\mu = \begin{bmatrix}0\\\sqrt{1-y^2}\\y\\0\end{bmatrix}. \tag{59}$$

In this way, their Lorentz invariants become trivial,

$$d^2 = r^2 = r'^2 = 1\,, \qquad d\cdot r = d\cdot r' = 0\,, \qquad r\cdot r' = y\,, \tag{60}$$

and we have

$$\not{d} = \gamma^4\,, \qquad \not{r} = \gamma^3\,, \qquad \not{r}' = \sqrt{1-y^2}\,\gamma^2 + y\,\gamma^3\,. \tag{61}$$

Moreover, we can define transversely projected $\gamma-$matrices $\gamma_\perp^\mu$ by

$$\gamma_\perp^\mu = \gamma^\mu - d^\mu\,\not{d} - r^\mu\,\not{r}\,, \qquad (\gamma_\perp^\mu)' = \gamma^\mu - d^\mu\,\not{d} - r'^\mu\,\not{r}'\,, \tag{62}$$

which entails for example

$$\gamma_\perp^1 = \gamma^1\,, \qquad \gamma_\perp^2 = \gamma^2\,, \qquad \gamma_\perp^3 = \gamma_\perp^4 = 0\,. \tag{63}$$

In this way we can express the quark-photon vertex in the following basis:

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{12} a_j(k^2, z, Q^2)\, i\tau_j^\mu(k,Q) \tag{64}$$

with

$$\begin{aligned}
\tau_1^\mu &= \tfrac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,, & \tau_5^\mu &= r^\mu\,\mathbb{1}\,, & \tau_9^\mu &= d^\mu\,\mathbb{1}\,,\\
\tau_2^\mu &= \tfrac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,\not{d}\,, & \tau_6^\mu &= r^\mu\,\not{d}\,, & \tau_{10}^\mu &= d^\mu\,\not{d}\,,\\
\tau_3^\mu &= \tfrac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,\not{r}\,, & \tau_7^\mu &= r^\mu\,\not{r}\,, & \tau_{11}^\mu &= d^\mu\,\not{r}\,,\\
\tau_4^\mu &= \tfrac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,\not{r}\,\not{d}\,, & \tau_8^\mu &= r^\mu\,\not{r}\,\not{d}\,, & \tau_{12}^\mu &= d^\mu\,\not{r}\,\not{d}\,,
\end{aligned} \tag{65}$$

which looks a lot simpler than the previous one. In particular, the τ_i^μ do not depend on any kinematic variable in the system since they are constructed only from unit vectors. The conjugate basis elements according to Eq. (48) become

$$\begin{aligned}
\bar\tau_j^\mu &= \tau_j^\mu \qquad \dots\ j = 1,5,6,7,9,10,11\\
\bar\tau_j^\mu &= -\tau_j^\mu \qquad \dots\ j = 2,3,4,8,12
\end{aligned} \tag{66}$$

Then, using the relations

$$\not{d}^2 = \not{r}^2 = 1\,, \qquad \not{r}\,\not{d} = -\not{d}\,\not{r}\,, \qquad \gamma_\perp^\mu\,\not{d} = -\not{d}\,\gamma_\perp^\mu\,, \qquad \gamma_\perp^\mu\,\not{r} = -\not{r}\,\gamma_\perp^\mu\,, \qquad \gamma_\perp^\mu\,\gamma_\perp^\mu = 2 \tag{67}$$

you can verify that we have indeed arrived at an orthonormal basis with the orthonormality relation

$$\frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau_i^\mu\,\tau_j^\mu\right\} = \delta_{ij}\,. \tag{68}$$

Note that the '1' and '2' appearing in (67) are Dirac matrices, i.e. they are understood to be multiplied by a 4×4 unit matrix in Dirac space, hence the $1/4$ in the orthogonality relation.

The two decompositions (40) and (65) are completely equivalent, except that (65) no longer satisfies the symmetries of the original basis, which means that the a_j are no longer even functions in $\omega = \sqrt{k^2}\sqrt{Q^2}z$. In any case, one can work out the linear relations between the two sets of dressing functions; abbreviating $s = \sqrt{1-z^2}$, then for the g_j these are

$$g_1 = a_{10} - \frac{z}{s}\,a_{11}\,, \qquad g_2 = \frac{1}{k^2\,zs}\,a_{11}\,, \qquad g_3 = -\frac{i}{kz}\,a_9\,, \qquad g_4 = \frac{i}{k^2\,Q\,zs}\,a_{12}\,. \tag{69}$$

One can show that these are consistent with the WTI: If we evaluate Eq. (44) using the decomposition (65), then only the longitudinal tensors $\tau_{9\dots12}^\mu$ survive, which determines the dressing functions $a_{9\dots12}$. Plugging them into the expressions above, the resulting g_i are identical to those in Eq. (45). The basis transformations for the f_j, on the other hand, are given by

$$\begin{aligned}
f_1 &= \frac{1}{Q^2\,s^2}\left(\frac{a_1}{\sqrt{2}} + zs\,(a_6+a_{11}) - z^2\,a_7 - s^2\,a_{10}\right), & f_5 &= -\frac{i}{k\,Q^2\,s}\left(a_5 - \frac{s}{z}\,a_9\right),\\
f_2 &= -\frac{i}{k^2\,Q^3\,s^2}\left(\frac{a_2}{\sqrt{2}} - a_8 + \frac{s}{z}\left(\frac{a_3}{\sqrt{2}} + a_{12}\right)\right), & f_6 &= -\frac{1}{k^2\,Q^2\,s^2}\left(\frac{a_1}{\sqrt{2}} - a_7 + \frac{s}{z}\,a_{11}\right),\\
f_3 &= \frac{i}{\sqrt{2}\,Q}\left(-a_2 + \frac{z}{s}\,a_3\right), & f_7 &= -\frac{1}{k^2\,Q^2\,s^2}\left(\frac{a_1}{\sqrt{2}} - a_7 + \frac{s}{z}\,a_6\right),\\
f_4 &= \frac{1}{\sqrt{2}\,k\,Q\,s}\,a_4\,, & f_8 &= \frac{i}{k^2\,Q\,s^2}\left(\frac{a_2}{\sqrt{2}} - a_8\right).
\end{aligned} \tag{70}$$

Before applying all this to the BSE (52), let us employ another convenient trick. Let us define

$$\Psi^\mu(k,Q) = S(k_+)\,\Gamma^\mu(k,Q)\,S(k_-) \tag{71}$$

as the quark-photon vertex with external quark propagators attached. Since this quantity has the same structural form as the vertex itself (4×4 Dirac matrix, one Lorentz index μ, two momenta k and Q), it must have the same basis decomposition (64) except with different dressing functions; let us call them b_j:

$$\Psi^\mu(k,Q) = \sum_{j=1}^{12} b_j(k^2, z, Q^2)\, i\tau_j^\mu(k,Q)\,. \tag{72}$$

In this way, we can split the BSE into a two-step process:

$$\begin{aligned}\Gamma^\mu(k,Q) &= Z_2\, i\gamma^\mu - \int_{k'} g(l^2)\, T_l^{\alpha\beta}\, \gamma^\alpha\, \Psi^\mu(k',Q)\, \gamma^\beta\,, \\ \Psi^\mu(k,Q) &= S(k_+)\, \Gamma^\mu(k,Q)\, S(k_-)\,,\end{aligned} \tag{73}$$

which avoids the need for evaluating the quark propagators inside the momentum loop and therefore saves some CPU time.

If we now insert the vertex decomposition (64) and employ the orthonormality relation (68), we arrive at purely scalar equations where all Lorentz and Dirac indices have disappeared,

$$\begin{aligned}a_i(k^2,z,Q^2) &= Z_2\, a_i^0 - \sum_{j=1}^{12} \int_{k'} g(l^2)\, K_{ij}(k^2,k'^2,z,z',y,Q^2)\, b_j(k'^2,z',Q^2)\,, \\ b_i(k^2,z,Q^2) &= \sum_{j=1}^{12} G_{ij}(k^2,z,Q^2)\, a_j(k^2,z,Q^2)\,,\end{aligned} \tag{74}$$

where $a_i^0 = \frac{1}{4}\,\mathrm{Tr}\,\{\bar\tau_i^\mu\,\gamma^\mu\}$ and the kernel matrix and propagator matrix are given by

$$\begin{aligned}K_{ij}(k^2,k'^2,z,z',y,Q^2) &= T_l^{\alpha\beta}\,\frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau_i^\mu(k,Q)\,\gamma^\alpha\,\tau_j^\mu(k',Q)\,\gamma^\beta\right\}, \\ G_{ij}(k^2,z,Q^2) &= \frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau_i^\mu(k,Q)\,S(k_+)\,\tau_j^\mu(k,Q)\,S(k_-)\right\}.\end{aligned} \tag{75}$$

Now you can see the *power* of using orthonormal bases: Had we used the original basis (40), the kernel matrix K_{ij} would be a complicated 12×12 matrix which depends on six variables. However, due to the simplicity of the τ_i^μ it simplifies dramatically:

- Because $d\cdot r = d\cdot r' = d\cdot\gamma_\perp = d\cdot\gamma_\perp' = 0$, the equations for the tensors $\tau_{1\ldots8}^\mu$ decouple completely from those for the $\tau_{9\ldots12}^\mu$. That is, both K_{ij} and G_{ij} fall apart into two non-interacting blocks,

$$K = \left(\begin{array}{c|c} 8\times8 & 0 \\ \hline 0 & 4\times4 \end{array}\right), \qquad G = \left(\begin{array}{c|c} 8\times8 & 0 \\ \hline 0 & 4\times4 \end{array}\right), \tag{76}$$

  so that the equations for $a_{1\ldots8}$ and those for $a_{9\ldots12}$ can be solved independently. The underlying reason is the WTI in Eq. (44), which says that the longitudinal part of the vertex defined by the $a_{9\ldots12}$ is completely determined by the quark propagator and decoupled from the dynamics.

- Because of $r\cdot\gamma_\perp = 0$, for the propagator matrix also the blocks for $a_{1\ldots4}$ and $a_{5\ldots8}$ decouple. In addition, because $r^2 = d^2 = 1$, the two lower blocks are identical:

$$G = \left(\begin{array}{c|c|c} G_{4\times4}^{(1)} & 0 & 0 \\ \hline 0 & G_{4\times4}^{(2)} & 0 \\ \hline 0 & 0 & G_{4\times4}^{(2)} \end{array}\right). \tag{77}$$

  This does not happen for the kernel because

$$r\cdot r' \neq 0, \qquad r\cdot\gamma_\perp' \neq 0\,, \qquad r'\cdot\gamma_\perp \neq 0\,, \tag{78}$$

  so the equations for $a_{1\ldots4}$ and $a_{5\ldots8}$ do not decouple after all. However, they are also not overly complicated because the $\tau_i^\mu(k,Q)$ do not depend on any kinematic variable, and the only kinematic variable that enters in the $\tau_i^\mu(k',Q)$ is y. Thus, the trace on the r.h.s. of the kernel in Eq. (75) can only depend on y and nothing else.

- From the definition of charge conjugation (48) one can show that the matrix G must be symmetric.

- In addition, the squared photon momentum Q^2 is an external variable, i.e., the equations for the $a_i(k^2,z,Q^2)$ are solved for each Q^2 separately. The kernel K also does not depend on Q^2 (which is a consequence of employing a rainbow-ladder interaction, because the gluon momentum $l = k - k'$ does not depend on Q) and Q^2 only enters in the propagator matrix G.

With this the ingredients of the BSE (74) are completely specified. For the integral $\int_{k'}$ in Eq. (20), the innermost integration over ϕ is trivial. Moreover, the amplitudes a_j and b_j in Eq. (74) do not depend on the variable y, so one can integrate over y right away to obtain $K'_{ij} = \int dy\, g(l^2)\, K_{ij}$. The strategy to solve the BSE is then as follows:

- Compute the kernel K'_{ij} by integrating over y.
- Loop over Q^2, since we want to solve the BSE for each Q^2.
- Solve the transverse ($a_{1\ldots8}$) and longitudinal ($a_{9\ldots12}$) equations separately for each Q^2. In each case, start with some initial guess for the a_i, calculate the b_i by applying the propagator matrix, and determine again the a_i by applying the kernel and integrating over the momentum. Proceed until converged.
- Convert the a_i into the g_j and f_j from Eq. (39) by applying the formulas (69–70).

Explicit traces

While it is straightforward to compute the traces (75) numerically in the code, an alternative is to work them out explicitly in advance. In principle this is not necessary, but especially with the orthonormal basis the resulting expressions are quite manageable. The inhomogeneous term in front of the BSE reads

$$a_i^0 = \frac{1}{4}\,\mathrm{Tr}\,\{\bar\tau_i^\mu\,\gamma^\mu\} = \begin{cases} \sqrt{2} & i=1 \\ 1 & i=7,10 \\ 0 & \text{else.} \end{cases} \tag{79}$$

From Eq. (57) the squared gluon momentum is

$$l^2 = (k-k')^2 = k^2 + k'^2 - 2k\cdot k' = k^2 + k'^2 - 2kk'\,(zz' + y\sqrt{1-z^2}\sqrt{1-z'^2})\,. \tag{80}$$

To make the following expressions more compact, we define

$$\begin{aligned} u &= k\sqrt{1-z^2}\,, \\ u' &= k'\sqrt{1-z'^2}\,, \end{aligned} \qquad V = \frac{kz-k'z'}{l^2}\,, \qquad w = \frac{u^2}{l^2}\,, \qquad w' = \frac{u'^2}{l^2}\,, \qquad X = \frac{uu'}{l^2}\,. \tag{81}$$

The entries of the kernel matrix then become

$$\begin{aligned} K_{11} &= -\tfrac{1+y^2}{2} - y\,(1-y^2)X\,, & K_{16} &= \sqrt{2}\,(1-y^2)\,u'V\,, \\ K_{22} &= -\tfrac{1+y^2}{2}\,(1-2l^2V^2) + y\,(1-y^2)X\,, & K_{61} &= -\sqrt{2}\,(1-y^2)\,uV\,, \\ K_{33} &= y(1-2l^2V^2) - (1-y^2)X\,, & K_{17} &= -\tfrac{1-y^2}{\sqrt{2}}\,(1+2w'-2yX)\,, \\ K_{44} &= y + (1-y^2)X\,, & K_{71} &= -\tfrac{1-y^2}{\sqrt{2}}\,(1+2w-2yX)\,, \\ K_{55} &= 3y\,, & K_{23} &= \big(2yu - (1+y^2)\,u'\big)\,V\,, \\ K_{66} &= -y\,(1+2l^2V^2)\,, & K_{32} &= -\big(2yu' - (1+y^2)\,u\big)\,V\,, \\ K_{77} &= -y^2(3-2l^2V^2) + 2y\,(1-y^2)X\,, & K_{67} &= 2y\,(u'-yu)\,V\,, \\ K_{88} &= y^2 - 2y\,(1-y^2)X\,, & K_{76} &= -2y\,(u-yu')\,V\,, \end{aligned} \tag{82}$$

together with

$$\begin{aligned} K_{28} &= K_{71} + \sqrt{2}\,(1-y^2)\,, \\ K_{82} &= K_{17} + \sqrt{2}\,(1-y^2)\,, \\ K_{38} &= -K_{61}\,, \\ K_{83} &= -K_{16}\,, \end{aligned} \qquad \begin{bmatrix} K_{99} \\ K_{10,10} \\ K_{10,11} \\ K_{11,10} \\ K_{11,11} \\ K_{12,12} \end{bmatrix} = \frac{1}{y} \begin{bmatrix} K_{55} \\ K_{66} \\ K_{67} \\ K_{76} \\ K_{77} \\ K_{88} \end{bmatrix} \tag{83}$$

and all other elements zero.

Concerning the propagator matrix, from the general decomposition of the quark propagator

$$S(k) = \frac{1}{A(k^2)}\,\frac{-i\not{k} + M(k^2)}{k^2 + M(k^2)^2} = \sigma_v(k^2)\,\big(-i\not{k} + M(k^2)\big) \tag{84}$$

and the appearance of $S(k_+)$ and $S(k_-)$ in Eq. (75) it is clear that there will be a common factor $\sigma_v(k_+^2)\,\sigma_v(k_-^2)$ in

front, where the rest can only depend on the mass function, i.e.

$$G_{ij}(k^2, z, Q^2) = \sigma_v(k_+^2)\,\sigma_v(k_-^2)\,\widetilde{G}_{ij}(k^2, z, Q^2)\,. \tag{85}$$

In analogy to Eq. (46), we define

$$\Sigma_M = \frac{M(k_+^2) + M(k_-^2)}{2}\,, \qquad \Delta_M = \frac{M(k_+^2) - M(k_-^2)}{k_+^2 - k_-^2}\,, \qquad \overline{M}^2 = M(k_+^2)\,M(k_-^2)\,. \tag{86}$$

For the first 4×4 block, the explicit calculation gives

$$\begin{aligned}
\widetilde{G}_{11} &= \overline{M}^2 + k^2 - \tfrac{Q^2}{4}\,, & \widetilde{G}_{12} &= iQ\,(\Sigma_M - 2k^2 z^2 \Delta_M)\,,\\
\widetilde{G}_{22} &= \overline{M}^2 - (1-2z^2)\,k^2 - \tfrac{Q^2}{4}\,, & \widetilde{G}_{13} &= -2ik^2\,Q\,z\sqrt{1-z^2}\,\Delta_M\,,\\
\widetilde{G}_{33} &= \overline{M}^2 + (1-2z^2)\,k^2 + \tfrac{Q^2}{4}\,, & \widetilde{G}_{14} &= -k\,Q\sqrt{1-z^2}\,,\\
\widetilde{G}_{44} &= \overline{M}^2 - k^2 + \tfrac{Q^2}{4}\,, & \widetilde{G}_{23} &= 2k^2 z\sqrt{1-z^2}\,,\\
& & \widetilde{G}_{24} &= 2ik\sqrt{1-z^2}\,\Sigma_M,\,,\\
& & \widetilde{G}_{34} &= ikz\,(Q^2\Delta_M - 2\Sigma_M)\,,
\end{aligned} \tag{87}$$

where the remaining entries are determined from the fact that G is symmetric, i.e., $G_{21} = G_{12}$ etc. The remaining two blocks can be reconstructed from the first block:

$$\begin{bmatrix} G_{55}\\ G_{56}\\ G_{66}\\ G_{77}\\ G_{78}\\ G_{88} \end{bmatrix} = \begin{bmatrix} G_{99}\\ G_{9,10}\\ G_{10,10}\\ G_{11,11}\\ G_{11,12}\\ G_{12,12} \end{bmatrix} = \begin{bmatrix} G_{44}\\ G_{34}\\ G_{33}\\ G_{22}\\ G_{12}\\ G_{11} \end{bmatrix}, \qquad \begin{bmatrix} G_{57}\\ G_{58}\\ G_{67}\\ G_{68} \end{bmatrix} = \begin{bmatrix} G_{9,11}\\ G_{9,12}\\ G_{10,11}\\ G_{10,12} \end{bmatrix} = - \begin{bmatrix} G_{24}\\ G_{14}\\ G_{23}\\ G_{13} \end{bmatrix}. \tag{88}$$

Chebyshev moments

In practice it is also useful to employ a Chebyshev expansion for the angular dependence in the variable z:

$$f(z) = \sum_{n=0}^{\infty} f_n\,U_n(z), \qquad f_n = \frac{2}{\pi}\int_{-1}^{1} dz\,\sqrt{1-z^2}\,U_n^*(z)\,f(z)\,, \qquad \frac{2}{\pi}\int_{-1}^{1} dz\,\sqrt{1-z^2}\,U_m^*(z)\,U_n(z) = \delta_{mn}\,. \tag{89}$$

The Chebyshev polynomials of the second kind $U_n(z)$ are given by

$$U_n(z) := \frac{\sin\left[(n+1)\,\arccos z\right]}{\sqrt{1-z^2}} = 2^n \prod_{k=1}^{n}\left[z - \cos\frac{k\,\pi}{n+1}\right], \quad (n \geq 0) \tag{90}$$

and can also be obtained from the recurrence relation

$$U_0(z) = 1\,, \quad U_1(z) = 2z\,, \quad U_n(z) = 2z\,U_{n-1}(z) - U_{n-2}(z)\,. \tag{91}$$

The first few Chebyshev polynomials ($n \geq 0$) are $U_n(z) = \big\{1,\ 2z,\ 4z^2-1,\ 8z^3-4z,\ 16z^4-12z^2+1,\ \ldots\big\}$. In fact, the dependence of the f_j and g_j on z is usually so weak that the zeroth Chebyshev moment is sufficient:

$$f_j^0(k^2, Q^2) = \frac{2}{\pi}\int_{-1}^{1} dz\sqrt{1-z^2}\,f_j(k^2, \omega, Q^2)\,. \tag{92}$$

If we apply this to the BSE (74) and write

$$a_i(k^2, z, Q^2) = \sum_m a_i^m(k^2, Q^2)\,U_m(z)\,, \qquad b_i(k^2, z, Q^2) = \sum_m b_i^m(k^2, Q^2)\,U_m(z)\,, \tag{93}$$

we arrive at the new equations for the Chebyshev moments:

$$\begin{aligned} a_i^m(k^2,Q^2) &= Z_2\, a_i^0\, \delta_{m0} + \sum_{jn} \int_0^{L^2} dk'^2\, K_{ij}^{mn}(k^2,k'^2)\, b_j^n(k'^2,Q^2)\,, \\ b_i^m(k^2,Q^2) &= \sum_{jn} G_{ij}^{mn}(k^2,Q^2)\, a_j^n(k^2,Q^2)\,, \end{aligned} \tag{94}$$

which have now obtained a very compact form. The Chebyshev moments of the kernel and propagator matrix are

$$\begin{aligned} K_{ij}^{mn}(k^2,k'^2) &= \frac{1}{(2\pi)^3}\,\frac{k'^2}{2}\,\frac{2}{\pi}\int dz\,\sqrt{1-z^2}\,U_m^*(z)\int dz'\,\sqrt{1-z'^2}\,U_n(z') \\ &\quad\times\int dy\, g(l^2)\, K_{ij}(k^2,k'^2,z,z',y)\,, \\ G_{ij}^{mn}(k^2,Q^2) &= \frac{2}{\pi}\int dz\,\sqrt{1-z^2}\,U_m^*(z)\,U_n(z)\,G_{ij}(k^2,z,Q^2)\,. \end{aligned} \tag{95}$$

Improving the accuracy

Although the orthonormal basis is simple to implement, it has a disadvantage: While the g_i and f_i only show a weak dependence in the angular variable z (remember $\omega = \sqrt{k^2}\sqrt{Q^2}\,z$), the a_i and b_i pick up a strong z dependence by Eqs. (69–70) through factors $1/\sqrt{1-z^2}$ or $1/(1-z^2)$, which diverge at $z \to \pm 1$, and this makes a polynomial expansion in z fairly difficult. For this reason, the numerical results would be actually more accurate if we solved the BSE directly for the g_i and f_i, however at the price that we would need a lot more CPU time to do so.

There is, however, a simple way to implement the best of both worlds. Suppose we take the basis (40) and divide out all factors of k^2 and Q^2, such that the remaining basis elements $t_i'^{\mu}$ only depend on z (cf. Eq. (57)). This is equivalent to redefining the g_i and f_i in terms of functions a_i':

$$\begin{aligned} g_1 &= a_9'\,, & g_2 &= \frac{a_{10}'}{k^2}\,, & g_3 &= \frac{a_{11}'}{k}\,, & g_4 &= \frac{a_{12}'}{Q\,k^2}\,, \\ f_1 &= \frac{a_1'}{Q^2}\,, & f_2 &= \frac{a_2'}{Q^3\,k^2}\,, & f_3 &= \frac{a_3'}{Q}\,, & f_4 &= \frac{a_4'}{Q\,k}\,, \\ f_5 &= \frac{a_5'}{Q^2\,k}\,, & f_6 &= \frac{a_6'}{Q^2\,k^2}\,, & f_7 &= \frac{a_7'}{Q^2\,k^2}\,, & f_8 &= \frac{a_8'}{Q\,k^2}\,. \end{aligned} \tag{96}$$

Then, the dressing functions a_i in Eq. (64) are related to the a_i' by a 12×12 transformation matrix $U(z)$:

$$a_i = \sum_{j=1}^{12} U_{ij}\, a_j'\,, \qquad U_{ij} = \frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau_i^{\mu}\, t'^{\mu}_j\right\}\,. \tag{97}$$

Applied to the BSE (74), changing the basis to go from $a_i \to a_i'$ merely amounts to replacing

$$a_i^0 \;\to U(z)^{-1}_{ij}\, a_j^0\,, \qquad K \;\to\; U(z)^{-1}\, K\, U(z')\,, \qquad G \;\to\; U(z)^{-1}\, G\, U(z)\,, \tag{98}$$

where K is the kernel in Eqs. (82–83) and G the propagator matrix in Eqs. (87–88). $U(z)$ only depends on the variable z and is given by

$$\begin{aligned} U_{11} = U_{19} &= \sqrt{2}\,, & U_{71} = U_{79} = U_{10,9} &= 1\,, \\ U_{17} &= \sqrt{2}\,z^2\,, & U_{76} = U_{7,10} &= s^2\,, \\ U_{22} = U_{28} = U_{2,12} &= \sqrt{2}\,iz^2\,, & U_{77} = U_{10,10} &= z^2\,, \\ U_{23} &= \sqrt{2}\,i\,, & U_{82} = U_{8,12} &= iz^2\,, \\ U_{32} = U_{38} = U_{3,12} &= \sqrt{2}\,isz\,, & U_{83} = U_{88} &= i\,, \\ U_{44} &= \sqrt{2}s\,, & U_{9,11} &= iz\,, \\ U_{55} = U_{5,11} &= is\,, & U_{12,12} &= -isz\,, \\ U_{66} = -U_{67} = U_{6,10} = U_{11,10} &= sz\,, \end{aligned} \tag{99}$$

with $s = \sqrt{1-z^2}$. Note that the new K has $U(z')$ on the right, i.e. one must replace $z \to z'$ and $s \to s' = \sqrt{1-z'^2}$. (G does not depend on z, here it is $U(z)$ on the right.)

The inverse of the matrix can be calculated numerically, but since it is also very simple it is faster to enter by hand. Writing $W = U^{-1}$, it reads

$$\begin{aligned}
W_{11} = -W_{61} = -W_{71} &= \tfrac{1}{\sqrt{2}\,s^2}\,, \\
W_{16} = W_{1,11} = -W_{9,11} &= \tfrac{z}{s}\,, \\
W_{17} &= -\tfrac{z^2}{s^2}\,, \\
W_{9,10} = -W_{1,10} &= 1\,, \\
W_{82} = -W_{22} &= \tfrac{i}{\sqrt{2}\,s^2}\,, \\
W_{23} &= -\tfrac{i}{\sqrt{2}\,sz}\,, \\
W_{28} = -W_{88} &= \tfrac{i}{s^2}\,,
\end{aligned}
\qquad
\begin{aligned}
W_{12,12} = -W_{2,12} &= \tfrac{i}{sz}\,, \\
W_{10,11} = -W_{6,11} = -W_{76} &= \tfrac{1}{sz}\,, \\
W_{32} &= -\tfrac{i}{\sqrt{2}}\,, \\
W_{33} &= \tfrac{iz}{\sqrt{2}\,s}\,, \\
W_{44} &= \tfrac{1}{\sqrt{2}\,s}\,, \\
W_{55} &= -\tfrac{i}{s}\,, \\
W_{59} = -W_{11,9} &= \tfrac{i}{z}\,, \\
W_{67} = W_{77} &= \tfrac{1}{s^2}\,.
\end{aligned}
\tag{100}$$

Note that U is complex, so the new K and G will be complex as well. A simple check after implementing U and U^{-1} to make sure that everything is correct is to test if $UU^{-1} = \mathbb{1}$.

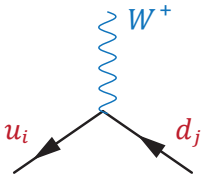
After this the BSE solution works in the same way as before, i.e., nothing else needs to be changed with one exception: the new K and G no longer decouple into 8×8 and 4×4 blocks, so one needs to solve the BSE as a 12×12 system. In the end, once the system has converged in terms of the a_i', we must reinstate the Q^2 and k^2 factors to obtain the actual dressing functions g_i and f_i according to Eq. (96) (whereas the basis transformations (69–70) are no longer needed).

6 Including weak interactions

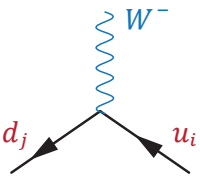
Now after this warmup, let's talk business: How do we generalize the BSE from the previous section to also include W-boson exchanges and arrive at the coupled system (11)?

Quark-photon vertex

To begin with, let us consider again the quark-photon vertex. Each entry K in Eq. (11) has the structure of Eq. (52), so for the gluon exchanges we don't need to add anything except enlarging the matrix. For the W-boson exchanges, we need to know the Feynman rules for the W boson coupling to quarks. These are given by



$$\frac{ig}{\sqrt{2}}\gamma^\mu P_L\,(V_{\text{CKM}})_{ij}\,,$$



$$\frac{ig}{\sqrt{2}}\gamma^\mu P_L\,(V_{\text{CKM}}^*)_{ij}\,. \tag{101}$$

Here, u_i stands for u-type quarks (u,c,t) and d_j for d-type quarks (d,s,b). g is not the QCD coupling from earlier but rather the $SU(2)$ weak isospin coupling, which is related to the electric charge e and the Weinberg angle θ_W and thus to the W- and Z-boson masses:

$$g = \frac{e}{\sin\theta_W}\,, \qquad \cos\theta_W = \frac{m_W}{m_Z}\,, \qquad e = \sqrt{4\pi\alpha_{\text{QED}}}\,. \tag{102}$$

Using

$$m_W = 80.4\,\text{GeV}\,, \qquad m_Z = 91.2\,\text{GeV}\,, \qquad \alpha_{\text{QED}} = \frac{1}{137.036} \tag{103}$$

one can extract g. The chiral projectors P_L and P_R are defined by

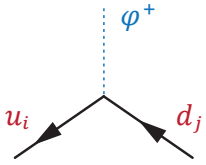
$$P_L = \frac{\mathbb{1} - \gamma_5}{2}\,, \qquad P_R = \frac{\mathbb{1} + \gamma_5}{2}\,, \tag{104}$$

which entails $\gamma^\mu P_L = P_R \gamma^\mu$. $(V_{\rm CKM})_{ij}$ are the CKM matrix elements for the generations $i, j = 1, 2, 3$ (see Appendix A for details). Furthermore, the W-boson propagator is given by

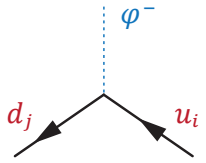
$$\frac{T_k^{\mu\nu}}{k^2 + m_W^2} + \frac{\xi_W L_k^{\mu\nu}}{k^2 + \xi_W m_W^2} \xrightarrow{\xi_W = 1} \frac{\delta^{\mu\nu}}{k^2 + m_W^2}\,, \tag{105}$$

with the transverse projector $T_k^{\mu\nu} = \delta^{\mu\nu} - k^\mu k^\nu / k^2$ and the longitudinal projector $L_k^{\mu\nu} = k^\mu k^\nu / k^2$. The gauge parameter ξ_W is arbitrary and should not affect observables. We will use the Feynman-'t Hooft gauge with $\xi_W = 1$ because here the propagator becomes maximally simple.

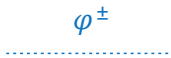
There is a catch, though, because depending on the choice of gauge unphysical Higgs degrees of freedom (the 'Goldstone modes') can propagate internally and we need to take them into account. These unphysical particles drop out in the unitary gauge ($\xi_W \to \infty$) usually discussed in the textbooks; however, in the unitary gauge the propagator does not fall off fast enough with k^2 and therefore its renormalization properties are not obvious. Therefore, to each $W^\pm$ exchange we should add an exchange of a $\varphi^\pm$ Goldstone particle, whose Feynman rules are given by



$$\frac{g}{\sqrt{2}} \frac{m_{ui} P_L - m_{dj} P_R}{m_W} (V_{\rm CKM})_{ij}\,,$$



$$\frac{g}{\sqrt{2}} \frac{m_{ui} P_R - m_{dj} P_L}{m_W} (V_{\rm CKM}^*)_{ij}\,, \tag{106}$$



$$\frac{1}{k^2 + \xi_W m_W^2} \xrightarrow{\xi_W = 1} \frac{1}{k^2 + m_W^2}\,.$$

With this we can write down the explicit expressions for the $\mathcal{M}$ kernels that appear in Eq. (11) and which are shown in Fig. 9. The top right block in Eq. (11) belongs to transitions from u to d-type quarks and the bottom left block to transitions from d to u-type quarks. Using the Feynman rules above and writing $V_{\rm CKM} = V$, we have

$$\begin{aligned} \mathcal{M}_{ab}^{cd}\,\Psi_{cd} \quad \to \quad -\frac{g^2}{2} V_{ac} V_{bd}^* \int_{k'} \frac{1}{l^2 + m_W^2} \Big[& \gamma^\alpha P_L\, \Psi_{cd}^\mu(k', Q) P_R\, \gamma^\alpha \\ & - \frac{1}{m_W^2} (m_a P_L - m_c P_R)\, \Psi_{cd}^\mu(k', Q)\, (m_b P_R - m_d P_L) \Big] \end{aligned} \tag{107}$$

for the top right block. The bottom left block is identical except for the replacement

$$V_{ac} V_{bd}^* \to V_{ca}^* V_{db}\,. \tag{108}$$

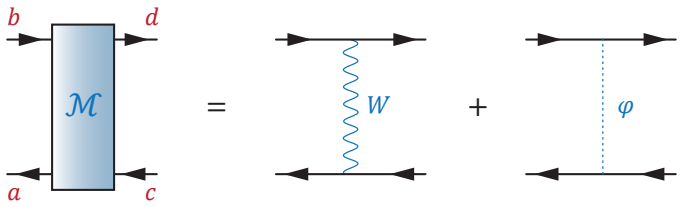


Figure 9: Off-diagonal kernels in the vertex BSE. ab and cd can be different flavors, but they must be of the same type and carry the same charge. For $ab \in \{uu, uc, cu, cc\}$ and $cd \in \{dd, ds, sd, ss\}$ these are the top right kernels in Eq. (11), for the bottom left kernels the situation is reversed.

From here it is straightforward to work out the traces in analogy to Eqs. (74)–(75). With the replacement

$$g(l^2) \to \widetilde{g}(l^2) = \frac{g^2}{2} \frac{1}{l^2 + m_W^2} \tag{109}$$

we can directly take over Eq. (74); the propagator matrix G_{ij} is the same as before and only the kernel is different:

$$\begin{aligned} K_{ij} &\to V_{ac} V_{bd}^* \, (\mathcal{M}_{ab}^{cd})_{ij} \quad \dots \text{ top right block,} \\ K_{ij} &\to V_{ca}^* V_{db} \, (\mathcal{M}_{ab}^{cd})_{ij} \quad \dots \text{ bottom left block,} \end{aligned} \tag{110}$$

where

$$\begin{aligned} (\mathcal{M}_{ab}^{cd})_{ij} = \frac{1}{4} \operatorname{Tr} \Big\{ \bar{\tau}_i^\mu(k,Q) \, \gamma^\alpha P_L \, \tau_j^\mu(k',Q) P_R \, \gamma^\beta \Big\} \\ - \frac{1}{4 m_W^2} \operatorname{Tr} \Big\{ \bar{\tau}_i^\mu(k,Q) \, (m_a P_L - m_c P_R) \, \tau_j^\mu(k',Q) (m_b P_R - m_d P_L) \Big\} \, . \end{aligned} \tag{111}$$

One can also work out the explicit traces in analogy to Eq. (82), which gives

$$\begin{aligned} &(\mathcal{M}_{ab}^{cd})_{11} = -c \frac{1+y^2}{2} \, , \qquad (\mathcal{M}_{ab}^{cd})_{44} = -(\mathcal{M}_{ab}^{cd})_{66} = -(\mathcal{M}_{ab}^{cd})_{11,11} = cy \, , \\ &(\mathcal{M}_{ab}^{cd})_{55} = 2dy \, , \qquad (\mathcal{M}_{ab}^{cd})_{77} = -cy^2 \, , \qquad (\mathcal{M}_{ab}^{cd})_{99} = 2d \, , \qquad (\mathcal{M}_{ab}^{cd})_{10,10} = -c \, , \\ &(\mathcal{M}_{ab}^{cd})_{17} = (\mathcal{M}_{ab}^{cd})_{71} = -c \frac{1-y^2}{\sqrt{2}} \, , \end{aligned} \tag{112}$$

with all other matrix elements zero, where we abbreviated

$$c = 1 - \frac{m_a \, m_b + m_c \, m_d}{m_W^2} \, , \qquad d = \frac{m_a \, m_d + m_b \, m_c}{m_W^2} \, . \tag{113}$$

With this you have all the tools to compute the quark-photon vertices including their flavor-changing components. For each component $\Gamma_{ab}^\mu(k,Q)$ there are twelve dressing functions as defined in Eq. (39), $g_j^{ab}(k^2,\omega,Q^2)$ and $f_j^{ab}(k^2,\omega,Q^2)$, so if we include two generations of quarks (up, down, strange, charm) this amounts to $12 \times 8 = 96$ dressing functions for the whole coupled system. To solve this system, we need to pay attention to the Feynman rule in Fig. 8: In the previous section we split off the factor eQ_f since we only considered one quark flavor. For the coupled system we can still drop the overall factor e, but we need to include the fermion charges Q_f in the tree-level terms $\Gamma_{ab}^{\mu,0}$ that enter in the equations.

All in all, the coupled BSEs (11) can be written in the following compact form:

$$\left[\begin{array}{c} a_i^{ab} \\ \hline a_i^{ab} \end{array}\right] = Z_2\, a_i^0 \left[\begin{array}{c} \frac{2}{3}\delta_{ab} \\ \hline -\frac{1}{3}\delta_{ab} \end{array}\right] - \int\limits_{k'} \left[\begin{array}{c|c} \delta_{ac}\delta_{bd}\, g(l^2)\, K_{ij} & V_{ac}V_{bd}^*\, \widetilde{g}(l^2)\, (\mathcal{M}_{ab}^{cd})_{ij} \\ \hline V_{ca}^* V_{db}\, \widetilde{g}(l^2)\, (\mathcal{M}_{ab}^{cd})_{ij} & \delta_{ac}\delta_{bd}\, g(l^2)\, K_{ij} \end{array}\right] \left[\begin{array}{c} b_j^{cd} \\ \hline b_j^{cd} \end{array}\right],$$
$$\left[\begin{array}{c} b_i^{ab} \\ \hline b_i^{ab} \end{array}\right] = \left[\begin{array}{c|c} G_{ij}^{ab} & 0 \\ \hline 0 & G_{ij}^{ab} \end{array}\right] \left[\begin{array}{c} a_j^{ab} \\ \hline a_j^{ab} \end{array}\right]. \tag{114}$$

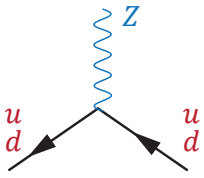
In the upper block, the superscript ab stands for $\{uu,\, uc,\, cu,\, cc\}$ and in the lower block it stands for $\{dd,\, ds,\, sd,\, ss\}$. This is the generalization of Eqs. (74)–(75). a_i^0, K_{ij} and G_{ij} are given in Eqs. (79) and below, and the $(\mathcal{M}_{ab}^{cd})_{ij}$ are given in Eq. (112). For the CKM matrix elements, see Appendix A. Concerning the G_{ij}^{ab}, the superscript ab only distinguishes the different quark masses that enter in the propagators (uu, uc, cu, ...), which means that in Eqs. (85–86) you need to distinguish

$$\text{Quark } a: \quad \sigma_v(k_+^2),\ M(k_+^2)\,, \qquad \text{Quark } b: \quad \sigma_v(k_-^2),\ M(k_-^2)\,, \tag{115}$$

which are obtained from different DSE solutions.

Z-boson vertex

For the Z-boson vertex things get more complicated. The reason is the structure of its tree-level Feynman rule, which reads



$$e\,\frac{I_3^f P_L - Q_f \sin^2\theta_W}{\sin\theta_W\, \cos\theta_W}\, Z_2\, i\gamma^\mu\,. \tag{116}$$

Here, I_3^f is the weak isospin 3-component of the quark and Q_f its charge:

$$u,\, c,\, t: \quad I_3^f = \frac{1}{2}\,, \quad Q_f = \frac{2}{3}\,; \qquad\qquad d,\, s,\, b: \quad I_3^f = -\frac{1}{2}\,, \quad Q_f = -\frac{1}{3}\,. \tag{117}$$

The difficulty comes from the left-handed projector $P_L = (\mathbb{1} - \gamma_5)/2$, which implies that the vertex no longer has a definite parity. For the quark-photon vertex, γ_5 terms did not appear because it had a definite parity, but for the $Z-$boson vertex this is no longer the case. As before, we split off the global factor e; using $\sin 2\theta_W = 2\sin\theta_W \cos\theta_W$, the tree-level vertex takes the form

$$\Gamma_0^\mu = \frac{I_3^f P_L - Q_f \sin^2\theta_W}{\sin\theta_W \cos\theta_W}\, Z_2\, i\gamma^\mu = \frac{I_3^f - 2Q_f \sin^2\theta_W}{\sin 2\theta_W}\, Z_2\, i\gamma^\mu + \frac{I_3^f}{\sin 2\theta_W}\, Z_2\, i\gamma^\mu\gamma_5\,. \tag{118}$$

Therefore, the tensor decomposition (39–40) is no longer sufficient and we also need to include γ_5 terms, and the resulting vertex has 24 tensors instead of 12.

Let's start with the orthonormal basis of Eq. (64). Its general decomposition reads

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{24} a_j(k^2, z, Q^2)\, i\tau_j^\mu(k,Q)\,, \qquad \tau_{13\ldots24}^\mu = \tau_{1\ldots12}^\mu\,\gamma_5\,, \qquad \bar{\tau}_{13\ldots24}^\mu = \gamma_5\,\bar{\tau}_{1\ldots12}^\mu\,. \tag{119}$$

Fortunately, the additional blocks for K, G and $\mathcal{M}$ are not overly complicated. The kernel K and

propagator matrix G generalize to

$$\mathbf{K} = \left(\begin{array}{c|c} K & 0 \\ \hline 0 & -K \end{array}\right), \qquad \mathbf{G} = \left(\begin{array}{c|c} G & 0 \\ \hline 0 & G' \end{array}\right), \tag{120}$$

where each block is a 12×12 matrix. The off-diagonal blocks are zero, so if we only considered QCD effects the positive- and negative-parity equations would simply decouple. (Note that this is all for one quark flavor; the full $\mathbf{K}$ and $\mathbf{G}$ matrices then enter as blocks in the flavor-coupled BSE (114)). G' has the same structure as Eq. (85),

$$G'_{ij}(k^2, z, Q^2) = \sigma_v(k_+^2)\, \sigma_v(k_-^2)\, \widetilde{G}'_{ij}(k^2, z, Q^2)\,, \tag{121}$$

except with

$$\begin{aligned}
\widetilde{G}'_{11} &= \overline{M}^2 - k^2 + \tfrac{Q^2}{4}\,, & \widetilde{G}'_{12} &= -ikz\,(Q^2\Delta_M - 2\Sigma_M)\,, \\
\widetilde{G}'_{22} &= \overline{M}^2 + (1-2z^2)\,k^2 + \tfrac{Q^2}{4}\,, & \widetilde{G}'_{13} &= 2ik\sqrt{1-z^2}\,\Sigma_M,\,, \\
\widetilde{G}'_{33} &= \overline{M}^2 - (1-2z^2)\,k^2 - \tfrac{Q^2}{4}\,, & \widetilde{G}'_{14} &= k\,Q\sqrt{1-z^2}\,, \\
\widetilde{G}'_{44} &= \overline{M}^2 + k^2 - \tfrac{Q^2}{4}\,, & \widetilde{G}'_{23} &= -2k^2 z\sqrt{1-z^2}\,, \\
& & \widetilde{G}'_{24} &= -2ik^2\,Q\,z\sqrt{1-z^2}\,\Delta_M\,, \\
& & \widetilde{G}'_{34} &= -iQ\,(\Sigma_M - 2k^2z^2\Delta_M)\,.
\end{aligned} \tag{122}$$

The remaining entries are still determined from the fact that G is symmetric, i.e., $G_{21} = G_{12}$ etc., and Eq. (88) still holds.

In turn, the tree-level vertex now contributes to more dressing functions in the inhomogeneous term. Eq. (79) still holds because the mixed-parity terms are zero, but because of Eq. (118) the inhomogeneous term in the equation generalizes to

$$Z_2\, a_i^0 \to \frac{Z_2}{\sin 2\theta_W} \begin{bmatrix} (I_3^f - 2Q_f \sin^2\theta_W)\, a_i^0 \\ I_3^f\, a_i^0 \end{bmatrix}, \tag{123}$$

with 12 entries in the upper block and 12 entries (those corresponding to γ_5) in the lower block. In each block there are only three non-zero entries, as given by Eq. (79), so this makes six non-zero entries in total.

The only non-zero off-diagonal blocks appear in the $\mathcal{M}$ kernels, because the W-boson exchanges couple together the γ^μ and $\gamma^\mu\gamma_5$ terms:

$$\mathbf{M}_{ab}^{cd} = \left(\begin{array}{c|c} \mathcal{M}_{ab}^{cd} & \widetilde{\mathcal{M}}_{ab}^{cd} \\ \hline -\widetilde{\mathcal{M}}_{ab}^{cd} & -\mathcal{M}_{ab}^{cd} \end{array}\right). \tag{124}$$

The $\widetilde{\mathcal{M}}_{ab}^{cd}$ are identical to the $\mathcal{M}_{ab}^{cd}$ in Eq. (112) except for the replacements

$$c \;\to\; \widetilde{c} = 1 - \frac{m_a\, m_b - m_c\, m_d}{m_W^2}\,, \qquad d \;\to\; \widetilde{d} = -\frac{m_a\, m_d - m_b\, m_c}{m_W^2}\,. \tag{125}$$

Using these relations, the BSE can be solved for the dressing functions a_j. What remains to be done is to establish the analogue of the basis (39) for the 24-dimensional case. To do so, we write

$$\begin{aligned}\Gamma^\mu(k,Q) = &\sum_{j=1}^{4} g_j(k^2,\omega,Q^2)\, iG_j^\mu(k,Q) + \sum_{j=1}^{8} f_j(k^2,\omega,Q^2)\, iT_j^\mu(k,Q) \\ &+ \sum_{j=5}^{12} g_j(k^2,\omega,Q^2)\, iG_j^\mu(k,Q)\gamma_5 + \sum_{j=9}^{12} f_j(k^2,\omega,Q^2)\, iT_j^\mu(k,Q)\gamma_5\,,\end{aligned} \tag{126}$$

where

$$\begin{aligned}
G_5^\mu &= \gamma^\mu\,, & G_9^\mu &= iQ^\mu\,, & T_9^\mu &= \omega\,\tfrac{i}{2}[\gamma^\mu, \not{Q}]\,,\\
G_6^\mu &= k^\mu \not{k}\,, & G_{10}^\mu &= Q^\mu \not{Q}\,, & T_{10}^\mu &= \tfrac{1}{6}[\gamma^\mu, \not{k}, \not{Q}]\,,\\
G_7^\mu &= i\omega\, k^\mu\,, & G_{11}^\mu &= \omega\, Q^\mu \not{k}\,, & T_{11}^\mu &= \omega\, t_{Qk}^{\mu\nu}\,\gamma^\nu\,,\\
G_8^\mu &= \tfrac{i}{2}[\gamma^\mu, \not{k}]\,, & G_{12}^\mu &= Q^\mu\,\tfrac{i}{2}[\not{k}, \not{Q}]\,, & T_{12}^\mu &= \omega\, t_{Qk}^{\mu\nu}\,\tfrac{i}{2}[\gamma^\nu, \not{k}]\,.
\end{aligned} \tag{127}$$

The basis transformations $a_j \to g_j, f_j$ (the generalizations of Eqs. (69)–(70)) are then given by

$$\begin{aligned}
g_5 &= \frac{1}{s^2}\left(\frac{a_{13}}{\sqrt{2}} + zs\,a_{18} - z^2\,a_{19}\right), & f_9 &= -\frac{i}{\sqrt{2}\,k\,Q^2\,z\,s}\,(s\,a_{14} - z\,a_{15})\,,\\
g_6 &= -\frac{1}{k^2\,s^2}\left(\frac{a_{13}}{\sqrt{2}} - a_{19}\right), & f_{10} &= \frac{1}{\sqrt{2}\,k\,Q\,s}\,a_{16},\\
g_7 &= -\frac{i}{k^2\,Q\,z\,s}\,a_{17}, & f_{11} &= -\frac{1}{k^2\,Q^2\,z\,s^2}\left(z\frac{a_{13}}{\sqrt{2}} + s\,a_{18} - z\,a_{19}\right),\\
g_8 &= -\frac{i}{k\,s^2}\left(\frac{z\,a_{14} + s\,a_{15}}{\sqrt{2}} - z\,a_{20}\right), & f_{12} &= \frac{i}{k^3\,Q^2\,z\,s^2}\left(\frac{a_{14}}{\sqrt{2}} - a_{20}\right),\\
g_9 &= \frac{i}{Q\,s}\,(z\,a_{17} - s\,a_{21})\,,\\
g_{10} &= -\frac{1}{Q^2\,s^2}\left(\frac{a_{13}}{\sqrt{2}} - z^2\,a_{19} - s^2\,a_{22} + zs\,(a_{18} + a_{23})\right),\\
g_{11} &= \frac{1}{k^2\,Q^2\,z\,s^2}\left(z\left(\frac{a_{13}}{\sqrt{2}} - a_{19}\right) + s\,a_{23}\right),\\
g_{12} &= -\frac{i}{k\,Q^2\,s^2}\left(z\left(\frac{a_{14}}{\sqrt{2}} - a_{20}\right) + s\left(\frac{a_{15}}{\sqrt{2}} + a_{24}\right)\right).
\end{aligned} \tag{128}$$

Improving the accuracy

In the final step, we would like to implement the analogues of Eq. (96) and below, i.e., solve the BSE directly in terms of functions a_j' which only differ from the g_j and f_j by factors of k and Q. Because the g_j, f_j depend only weakly on z, the same is true for the a_j' and thus we can get away with a small number of Chebyshev moments (or equivalently a small number of grid points in z).

For the quark-photon vertex with its 12 components you can directly take over the relations below Eq. (96) since this is all you need. For the quark-Z-boson vertex, however, there are twice as many tensors and so these relations need to be generalized. We employ the same strategy as before, i.e., we divide out factors of k and Q from the tensor

basis (127) to arrive at tensors $t'^{\mu}_{\ j}$ with respective dressing functions a'_j:

$$\begin{aligned}
g_5 &= a'_{13}\,, & g_6 &= \frac{a'_{14}}{k^2}\,, & g_7 &= \frac{a'_{15}}{Q\,k^2}\,, & g_8 &= \frac{a'_{16}}{k}\,,\\
g_9 &= \frac{a'_{17}}{Q}\,, & g_{10} &= \frac{a'_{18}}{Q^2}\,, & g_{11} &= \frac{a'_{19}}{Q^2\,k^2}\,, & g_{12} &= \frac{a'_{20}}{Q^2\,k}\,,\\
f_9 &= \frac{a'_{21}}{Q^2\,k}\,, & f_{10} &= \frac{a'_{22}}{Q\,k}\,, & f_{11} &= \frac{a'_{23}}{Q^2\,k^2}\,, & f_{12} &= \frac{a'_{24}}{Q^2\,k^3}\,.
\end{aligned} \tag{129}$$

Then, the dressing functions a_j in Eq. (119) are related to the a'_j by a 24×24 transformation matrix $\mathbf{U}(z)$:

$$a_i = \sum_{j=1}^{24} \mathbf{U}_{ij}\, a'_j\,, \qquad \mathbf{U}_{ij} = \frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau_i^{\mu}\, t'^{\mu}_{\ j}\right\}\,. \tag{130}$$

The off-diagonal blocks in $\mathbf{U}$ vanish due to the extra γ_5 that appears in the trace, therefore

$$\mathbf{U} = \begin{pmatrix} U & 0\\ 0 & U'\end{pmatrix} \qquad \Rightarrow \qquad \mathbf{W} = \mathbf{U}^{-1} = \begin{pmatrix} U^{-1} & 0\\ 0 & U'^{-1}\end{pmatrix} = \begin{pmatrix} W & 0\\ 0 & W'\end{pmatrix}. \tag{131}$$

The upper left block U is identical to Eq. (133), so what remains to be calculated is the additional lower right block U'. As before, changing the basis to go from $a_i \to a'_i$ merely amounts to replacing

$$a_i^0 \;\to \mathbf{W}(z)_{ij}\, a_j^0\,, \qquad \mathbf{K} \;\to\; \mathbf{W}(z)\,\mathbf{K}\,\mathbf{U}(z')\,, \qquad \mathbf{M} \;\to\; \mathbf{W}(z)\,\mathbf{M}\,\mathbf{U}(z')\,, \qquad \mathbf{G} \;\to\; \mathbf{W}(z)\,\mathbf{G}\,\mathbf{U}(z)\,. \tag{132}$$

$U'(z)$ is given by

$$\begin{aligned}
U'_{11} &= \sqrt{2}\,, & U'_{2,12} &= \sqrt{2}\,iz^3\,,\\
U'_{1,11} &= \sqrt{2}\,z^2\,, & U'_{34} &= \sqrt{2}\,is\,,\\
U'_{24} = U'_{29} &= \sqrt{2}\,iz\,, & U'_{3,12} &= \sqrt{2}\,isz^2\,,\\
U'_{84} = U'_{89} = U'_{8,12} &= iz\,, & U'_{4,10} &= \sqrt{2}\,s\,,\\
U'_{62} = U'_{11,2} = U'_{11,7} = -U'_{6,11} &= sz\,, & U'_{53} &= isz\,,\\
U'_{71} = U'_{10,1} = U'_{10,6} &= 1\,, & U'_{72} &= s^2\,,\\
U'_{7,11} = U'_{10,2} = U'_{10,7} &= z^2\,, & U'_{93} &= iz^2\,,\\
U'_{12,8} = -U'_{12,4} &= is\,, & U'_{95} &= i\,,
\end{aligned} \tag{133}$$

with $s = \sqrt{1-z^2}$. Note again that the new $\mathbf{K}$ and $\mathbf{M}$ have $\mathbf{U}(z')$ on the right, i.e. one must replace $z \to z'$ and $s \to s' = \sqrt{1-z'^2}$. $\mathbf{G}$ does not depend on z, here it is $U(z)$ on the right. The inverse matrix W' reads

$$\begin{aligned}
W_{11} = W_{71} = -W_{21} = -W_{61} &= \tfrac{1}{\sqrt{2}\,s^2}\,, & W_{35} &= -\tfrac{i}{sz}\,,\\
W_{16} = -W_{66} = -W_{6,11} &= \tfrac{z}{s}\,, & W_{55} &= \tfrac{iz}{s}\,,\\
W_{67} = -W_{17} &= \tfrac{z^2}{s^2}\,, & W_{59} &= -i\,,\\
W_{27} = W_{11,7} &= \tfrac{1}{s^2}\,, & W_{6,10} &= 1\,,\\
W_{42} = W_{82} &= -\tfrac{iz}{\sqrt{2}\,s^2}\,, & W_{77} &= -\tfrac{1}{s^2}\,,\\
W_{93} = -W_{43} = -W_{83} &= \tfrac{i}{\sqrt{2}\,s}\,, & W_{8,12} &= -\tfrac{i}{s}\,,\\
W_{48} = W_{88} &= \tfrac{iz}{s^2}\,, & W_{92} &= -\tfrac{i}{\sqrt{2}\,z}\,,\\
W_{7,11} = -W_{11,6} &= \tfrac{1}{sz}\,, & W_{10,4} &= \tfrac{1}{\sqrt{2}\,s}\,,\\
& & W_{11,1} &= -\tfrac{1}{\sqrt{2}\,s^2}\,,\\
& & W_{12,2} &= \tfrac{i}{\sqrt{2}\,s^2 z}\,,\\
& & W_{12,8} &= -\tfrac{i}{s^2 z}\,.
\end{aligned} \tag{134}$$

In the end, once the system has converged in terms of the a'_i, we must reinstate the Q^2 and k^2 factors to obtain the actual dressing functions g_i and f_i according to Eqs. (96) and (129).

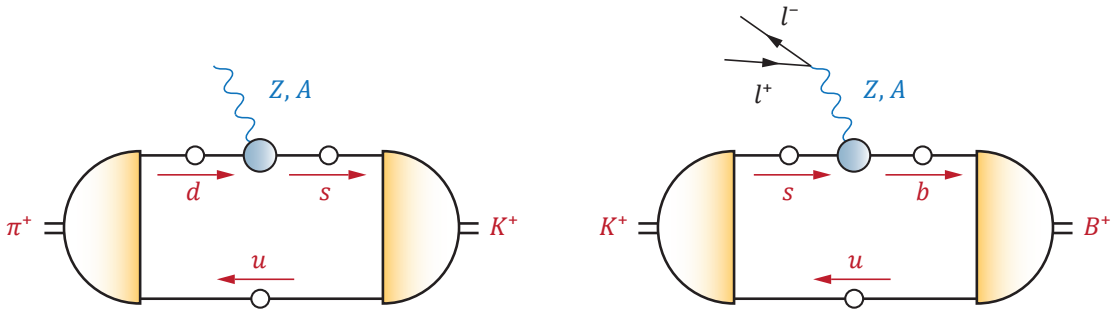


Figure 10: $K^+ \to \pi^+$ and $B^+ \to K^+$ transition matrix elements

7 Transition matrix elements and flavor anomalies

Ok! If we make it to this point by the end of the summer, you accomplished a lot: you calculated the nonperturbative quark-photon and quark-Z-boson vertex in terms of their dressing functions $f_j(k^2, \omega, Q^2)$ and $g_j(k^2, \omega, Q^2)$. Now, depending on how fast things go, we may consider to implement this in the calculation of an actual *observable*, like the $K \to \pi$ and $B \to K$ matrix elements in Fig. (10). The quantities to extract from there are the **transition form factors** $F_{K\to\pi}(Q^2)$ or $F_{B\to K}(Q^2)$, in analogy to the electromagnetic form factors for fermions in Eq. (35). Without internal W bosons, these form factors would be zero since a photon or Z boson cannot change the flavor, so they will be very small. Nevertheless, such transition form factors are just the information we need to pin down the QCD contributions to flavor anomalies.

As an example, the LHCb collaboration recently found 3.1σ evidence for the violation of lepton universality in $B^+ \to K^+ l^+ l^-$ decays [6]. In particular, the branching ratio

$$R_K = \frac{\mathcal{B}(B^+ \to K^+ \mu^+ \mu^-)}{\mathcal{B}(B^+ \to K^+ e^+ e^-)}\,, \tag{135}$$

slightly differs from 1, although in the SM it should be exactly 1 because electrons and muons have the same interaction strengths. In principle such a difference could be induced by new particles with new interactions beyond the SM. However, although QCD effects should drop out from this ratio, the electrons and muons have different masses and the form factor is therefore tested at different (timelike) values of Q^2, so there could be residual effects from QCD.

I will leave this part for later since it requires some additional information:

- We need the dressed quark propagators for up, down, strange, ... – which we have, since you already needed them to calculate the vertices.
- For the $s \to d$ transition we can use what we have, but for the $b \to s$ vertex you would need to include all three fermion generations to solve the BSE.
- We need the π, K, ... meson Bethe-Salpeter amplitudes – which you didn't calculate, but we have numerical results for them which you can use to calculate the matrix elements.
- The complication is that internally we need the quark propagators, the quark-photon vertex and the quark-Z-boson vertex for *complex* momenta, i.e., complex k^2, but so far you computed the vertex only for real and positive k^2. Although implementing complex k^2 is in principle straightforward, this goes beyond what we want to do here. Therefore, with the input we have we can obtain the transition form factor only for (unphysical) real and positive values of the incoming and outgoing momenta, $P_i^2 > 0$ and $P_f^2 > 0$, whereas we need them at the physical onshell values $P_i^2 = -m_K^2$ and $P_f^2 = -m_\pi^2$. One way to proceed would be to analytically continue the form factor using Padé approximants or the so-called Schlessinger-point method (SPM); this should not be difficult to implement but let's see if we get there.

A CKM matrix

The **CKM matrix** is a unitary $N \times N$ matrix and thus depends on N^2 real parameters. $2N-1$ of those are physically not significant, because one phase can be absorbed into each quark field but the matrix is independent of a common phase. Hence, the total number of independent variables is $N^2-(2N-1)=(N-1)^2$. Of these, $N(N-1)/2$ are rotation angles called quark mixing angles, and the remaining $(N-2)(N-1)/2$ are complex phases which cause CP violation.

For $N=2$ there is only one parameter, the **Cabbibo angle** θ_{12} which is a mixing angle between two generations of quarks. For $N=3$, which is the case of the Standard Model, we have

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}, \tag{136}$$

which can be expressed in terms of three mixing angles θ_{12}, θ_{23}, θ_{13} and one CP-violating phase δ. In the 'standard parametrization', the CKM matrix is written as:

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\, e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{i\delta} & c_{23}\, c_{13} \end{pmatrix} \end{aligned} \tag{137}$$

with $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$. In practice it turns out that

$$s_{13} \ll s_{23} \ll s_{12} \ll 1\,, \tag{138}$$

which motivates the **Wolfenstein parametrization** with four parameters λ, A, ρ and η:

$$s_{12} = \lambda\,, \qquad s_{23} = A\lambda^2\,, \qquad s_{13}\, e^{i\delta} = A\lambda^3(\rho + i\eta) = \frac{A\lambda^3(\bar\rho + i\bar\eta)\sqrt{1-A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1 - A^2\lambda^4(\bar\rho + i\bar\eta)]}\,. \tag{139}$$

With this the CKM matrix can be expanded in powers of λ:

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\,. \tag{140}$$

One can see that the dominant transitions are V_{ud}, V_{cs}, V_{tb}, followed by the subleading transitions V_{us} and V_{cd} which are of $\mathcal{O}(\lambda)$, and all further transitions are suppressed with at least $\mathcal{O}(\lambda^2)$.

The PDG values for the Wolfenstein parameters are (rounded)

$$\lambda = 0.225, \qquad A = 0.83, \qquad \bar\rho = 0.12, \qquad \bar\eta = 0.35\,, \tag{141}$$

from where the CKM elements can be reconstructed.

B Formulas

We use a **Euclidean metric** throughout all calculations. If you are used to Minkowski space (which is what you would learn in a relativistic QM or QFT course), this may need some getting used to. However, chances are that you are actually *not* yet all too familiar with it anyway, so we can just jump to Euclidean space directly because this is how most actual calculations are done in practice. (In case you already know a bit about QFT, then you may have heard that for practical calculations one has to do a 'Wick rotation' to Euclidean space, but one can equivalently start from a Euclidean QFT directly.)

The basic idea is to collect space and time into a four-vector, where the fourth component stands for imaginary time:

$$x^\mu = \begin{bmatrix} \boldsymbol{x} \\ ix_0 \end{bmatrix}, \qquad x_0 = t\,. \tag{142}$$

(We always use natural units where $c = \hbar = 1$.) In momentum space, for an onshell particle the fourth component then takes the meaning of imaginary energy:

$$p^\mu = \begin{bmatrix} \boldsymbol{p} \\ ip_0 \end{bmatrix}, \qquad p_0 = E_p = \sqrt{\boldsymbol{p}^2 + m^2} \qquad \Rightarrow \qquad p^2 = \sum_{\mu=1}^{4} p^\mu p^\mu = -m^2\,. \tag{143}$$

We follow the Einstein summation convention, i.e., two Lorentz indices are always summed over: $p^2 = p \cdot p = p^\mu p^\mu$, $p \cdot q = p^\mu q^\mu$ etc. There is also no distinction between upper and lower Lorentz indices since in Euclidean space all four directions are on equal footing.

As already mentioned earlier, it is convenient to express four-momenta through hyperspherical coordinates,

$$p^\mu = \sqrt{p^2} \begin{pmatrix} \sqrt{1-z^2}\,\sqrt{1-y^2}\,\sin\phi \\ \sqrt{1-z^2}\,\sqrt{1-y^2}\,\cos\phi \\ \sqrt{1-z^2}\;\; y \\ z \end{pmatrix} = \begin{pmatrix} \sin\psi\,\sin\theta\,\sin\phi \\ \sin\psi\,\sin\theta\,\cos\phi \\ \sin\psi\,\cos\theta \\ \cos\psi \end{pmatrix}, \tag{144}$$

which are the straightforward generalizations of spherical coordinates to four dimensions. The point is that in loop calculations in QFT the particles are not onshell but *virtual*, which means their $p^2 > 0$ is positive or 'spacelike' (hence the need for going to Euclidean space). Correspondingly, a four-momentum integration reads

$$\int\limits_p = \int \frac{d^4p}{(2\pi)^4} = \frac{1}{(2\pi)^4}\,\frac{1}{2} \int\limits_0^\infty dp^2\,p^2 \int\limits_{-1}^{1} dz\,\sqrt{1-z^2} \int\limits_{-1}^{1} dy \int\limits_0^{2\pi} d\phi\,, \tag{145}$$

where one can equivalently write

$$\frac{1}{2} \int\limits_0^\infty dp^2\,p^2 = \int\limits_0^\infty dp\,p^3\,. \tag{146}$$

If we want to compute things onshell (for 'real' particles), we must remember that in this case $p^2 < 0$ ('timelike'), which comes with additional complications because p^μ must have some imaginary components. For example, an onshell particle ($p^2 = -m^2$) in the rest frame ($\boldsymbol{p} = 0 \Rightarrow E_p = m$) has the four-vector

$$p^\mu = im \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}. \tag{147}$$

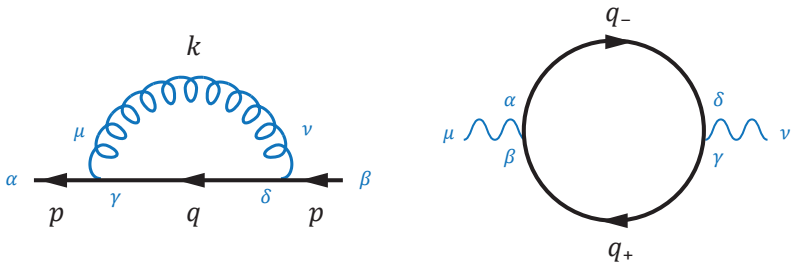


Figure 11: Lorentz and Dirac indices in momentum loops

We also frequently employ **gamma matrices**, which are needed for the description of fermions with spin 1/2 (such as quarks, electrons, muons, ...). These are 4×4 matrices, which in the standard representation have the form

$$\gamma^k = \begin{pmatrix} 0 & -i\tau_k \\ i\tau_k & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{148}$$

where $k = 1, 2, 3$ and the τ_k are the 2×2 Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{149}$$

The Euclidean gamma matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \qquad \gamma^\mu = (\gamma^\mu)^\dagger, \qquad \gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4, \tag{150}$$

where $\{A, B\} = AB + BA$ is the anticommutator. We also make frequent use of the Feynman slash notation:

$$\not{p} = p \cdot \gamma = p^\mu \gamma^\mu. \tag{151}$$

This entails $\not{p}^2 = p^2$ because

$$\not{p}^2 = p^\mu p^\nu \gamma^\mu \gamma^\nu = p^\mu p^\nu \left(\frac{1}{2} [\gamma^\mu, \gamma^\nu] + \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} \right) = 0 + p^\mu p^\nu \delta^{\mu\nu} = p^\mu p^\mu = p^2. \tag{152}$$

The point is that spin-1/2 particles are described by Dirac spinors with Dirac indices $\alpha = 1 \ldots 4$. As a consequence, a quantity with an incoming and outgoing quark leg (like the quark propagator, quark-photon vertex etc.) has two Dirac indices α, β and thus it must be a 4×4 matrix in Dirac space. The gamma matrices provide a complete basis in Dirac space. For example, the only possible basis elements for a spin-1/2 propagator, which depends on one momentum p^μ, are $\mathbb{1}$ and $\not{p}$. Hence there are *two* Lorentz-invariant dressing functions $A(p^2)$ and $M(p^2)$. The quark-photon vertex has an additional Lorentz index because the photon has spin 1, and in this case there are 12 possible tensors which can be expressed through any of the bases (40), (65) or linear combinations thereof.

When you draw a Feynman diagram, you write down all Dirac and Lorentz indices and follow the fermion lines from the tip of the 'spin arrow' to its end like in Fig. 11:

$$\begin{aligned} \int_q \gamma^\mu_{\alpha\gamma} S_{\gamma\delta}(q) \gamma^\nu_{\delta\beta} D^{\mu\nu}(k) &= \int_q \{\gamma^\mu S(q) \gamma^\nu\}_{\alpha\beta} D^{\mu\nu}(k), \\ \int_q \gamma^\mu_{\alpha\beta} S_{\beta\gamma}(q_+) \gamma^\nu_{\gamma\delta} S_{\delta\alpha}(q_-) &= \int_q \mathrm{Tr}\{\gamma^\mu S(q_+) \gamma^\nu S(q_-)\}. \end{aligned} \tag{153}$$

In this way, a closed fermion loop amounts to taking a Dirac trace. The traces of gamma matrices are easy to work out using the formulas

$$\begin{aligned} \operatorname{Tr} \mathbb{1} &= 4\,, \\ \operatorname{Tr} \gamma^\mu &= 0\,, \\ \tfrac{1}{4} \operatorname{Tr} \{\gamma^\mu \gamma^\nu\} &= \delta^{\mu\nu}\,, \\ \operatorname{Tr} \{\gamma^\mu \gamma^\nu \gamma^\rho\} &= 0\,, \\ \tfrac{1}{4} \operatorname{Tr} \{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} &= \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\rho\nu} - \delta^{\mu\rho}\delta^{\nu\sigma}\,, \end{aligned} \tag{154}$$

etc. In particular, the trace over an odd number of gamma matrices vanishes (we used this property several times in the previous sections). This also implies

$$\begin{aligned} \operatorname{Tr} \not{p} &= 0\,, \\ \tfrac{1}{4} \operatorname{Tr} \{\not{p}\, \not{q}\} &= p \cdot q\,, \\ \operatorname{Tr} \{\not{p}\, \not{q}\, \not{k}\} &= 0\,, \\ \tfrac{1}{4} \operatorname{Tr} \{\not{p}\, \not{q}\, \not{k}\, \not{l}\} &= (p \cdot q)(k \cdot l) + (p \cdot l)(k \cdot q) - (p \cdot k)(q \cdot l)\,. \end{aligned} \tag{155}$$

Using these formulas, we can do almost all calculations without the need for knowing the explicit form of the gamma matrices from Eq. (148).

Another matrix we defined in Sec. 5 is the charge conjugation matrix, which satisfies

$$C = \gamma^4 \gamma^2, \quad C^T = C^\dagger = C^{-1} = -C. \tag{156}$$

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